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SOME INTEGRALS INVOLVING GENERALIZED ASSOCIATED
LEGENDRE POLYNOMIALS

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ABSTRACT

Kharadze (1935) has given a generalization of Legendre polynomials by using the operator $D_k^k = \frac{d}{dx} \cdot \frac{1}{x^k - 2} \cdot \frac{d}{dx}$ in the Rodrigues formula. Using the same operator A. Sharma (1948) has defined the generalized Associated Legendre polynomials as

$$Q_{mk}^{nk}(z) = \frac{1}{k^{2m} 2m!} (1-z^k)^n D_k^{mk+nk} (1-z^k)^{2m}.$$

$$Q_{mk}^{nk+1}(z) = \frac{1}{k^{2m} 2m!} (1-z^k)^{n+1/k} D_k^{mk+nk+k-1} (1-z^k)^{2m},$$

In the present paper we have evaluated some integrals involving generalized Associated Legendre polynomials with generalized Hypergeometric function, and other functions which are expressible in the Hypergeometric form, such as, Bessel functions, product of two Bessel functions $J_\mu(x) J_\lambda(x)$, product of modified Bessel function and Bessel function $I_\mu(x) J_\mu(x)$, generalized Hermite polynomials defined by Chatterji (1955) $H_{mk}(z)$, product of Whittaker $M_{k,\mu}(x)$ function and Laguerre $L_n^\alpha(x)$ polynomials. We have also established two identities by using the relations [Watson, 1952, § 5.21 § 5.23] in which the Bessel functions has been expressed as an infinite series containing Bessel functions. The method employed is term by term integration.

Introduction : Kharadze [4] has given a generalization of Legendre Polynomials defined by

$$Q_{mk}(z) = \frac{1}{2^m 2m!} D_k^{mk} (z^k - 1)^{2m}$$

$$Q_{mk+1}(z) = \frac{1}{k^{2m+1} (2m+1)!} D_k^{mk+k-1} (z^k - 1)^{2m+1}$$

where $k \geq 2$ is a fixed natural number. $D_k^k = \frac{d}{dz} \frac{1}{z^{k-2}} \frac{d}{dz}$, $D_k^{k-1} = \frac{1}{z^{k-2}} \frac{d}{dz}$, D_k^{mk} means the operator D_k^k repeats m times. Using the same operator

Sharma [6] have given a generalization for Associated Legendre Polynomials as follows :

$$Q_{mk}^{nk}(z) = \frac{1}{k^{2m} 2m!} (1-z^k)^n D_k^{mk+nk} (1-z^k)^{2m}$$

$$Q_{mk}^{nk+1}(z) = \frac{1}{k^{2m} 2m!} (1-z^k)^{n+1/k} D_k^{mk+nk+k-1} (1-z^k)^{2m}$$

In this paper, we have evaluated some integrals involving generalised Associated Legendre Polynomials. The following two integrals

$$\int_0^1 x^{k\sigma} (1-x^k)^{-n} Q_{mk}^{nk}(x) dx, \text{Re } k\sigma > -1, m \geq n,$$

$$\int_0^1 x^{k\sigma} (1-x^k)^n Q_{mk}^{nk}(x) dx, \text{Re } k\sigma > -1, m \geq n$$

are fundamental in obtaining the results of this paper. The integrals evaluated are

$$(i) \int_0^1 x^{k\sigma} (1-x^k)^{-n} Q_{mk}^{nk}(x) J_s \left\{ 2(xt)^{k/2} \right\} dx, \text{Re } (k\sigma + sk/2) > -1,$$

$$(ii) \int_0^1 x^{k\sigma} (1-x^k)^n Q_{mk}^{nk}(x) J_s \left\{ 2(xt)^{k/2} \right\} dx, \text{Re } (k\sigma + sk/2) > -1,$$

$$(iii) \int_0^1 x^{k\sigma - k\lambda/2 - k\mu/2} (1-x^k)^{-n} Q_{mk}^{nk}(x) J_\mu \left\{ (xt)^{k/2} \right\} J_\lambda \left\{ (xt)^{k/2} \right\} dx,$$

$$\text{Re } k\sigma > -1,$$

$$(iv) \int_0^1 x^{k\sigma - k\lambda/2 - k\mu/2} (1-x^k)^n Q_{mk}^{nk}(x) J_\mu \left\{ (xt)^{k/2} \right\} J_\lambda \left\{ (xt)^{k/2} \right\} dx,$$

$$\text{Re } k\sigma > -1,$$

$$(v) \int_0^1 x^{k\sigma - k\mu/2} (1-x^k)^{-n} Q_{mk}^{nk}(x) I_\mu \left(x^{k/4} \right) J_\mu \left(x^{k/4} \right) dx,$$

$$\text{Re } k\sigma > -1,$$

$$(vi) \int_0^1 x^{k\sigma - k\mu/2} (1-x^k)^n Q_{mk}^{nk}(x) I_\mu \left(x^{k/4} \right) J_\mu \left(x^{k/4} \right) dx, \text{Re } k\sigma > -1,$$

$$(vii) \int_0^1 x^{k\sigma} (1-x^k)^{-n} Q_{mk}^{nk}(x) {}_pF_q \left[\begin{matrix} \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q \end{matrix} ; \pm (xt)^k \right] dx,$$

Re $k\sigma > -1$, and

$$(viii) \int_0^1 x^{k\sigma} (1-x^k)^n Q_{mk}^{nk}(x) {}_pF_q \left[\begin{matrix} \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q \end{matrix} ; \pm (xt)^k \right] dx,$$

Re $k\sigma > -1$.

The method employed is term by term integration.

1. Substituting the value of $Q_{mk}^{nk}(x)$ in the integral

$$\int_0^1 x^{k\sigma} (1-x^k)^{-n} Q_{mk}^{nk}(x) dx$$

and integrating by parts, we obtain

$$\begin{aligned} \int_0^1 x^{k\sigma} (1-x^k)^{-n} Q_{mk}^{kn}(x) dx \\ = \frac{k^{2n-1} \Gamma(\sigma+1) \Gamma(\sigma+1/k)}{\Gamma(\sigma+1-m-n) \Gamma(\sigma+1/k+m-n+1)} \end{aligned}$$

provided $k\sigma > -1$, m, n being integers.

Similarly substituting the value of $Q_{mk}^{nk}(x)$ in the integral

$$\int_0^1 x^{k\sigma} (1-x^k)^n Q_{mk}^{nk}(x) dx$$

and using Sharma's [6] identity

$$\begin{aligned} (1-z^k)^{2n} D_k^{m+k+nk} (1-z^k)^{2m} &= \frac{k^{4n} (m+n)! \Gamma(m+n+1/k)}{(m-n)! \Gamma(m-n+1/k)} \\ &\quad \cdot D_k^{(m-n)k} (1-z^k)^{2m} \end{aligned}$$

and integrating by parts, we obtain

$$\begin{aligned} \int_0^1 x^{k\sigma} (1-x^k)^n Q_{mk}^{nk}(x) dx \\ = \frac{k^{2n-1} \Gamma(m+n+1) \Gamma(m+n+1/k) \Gamma(\sigma+1) \Gamma(\sigma+1/k)}{\Gamma(m-n+1) \Gamma(m-n+1/k) \Gamma(\sigma+1-m+n) \Gamma(\sigma+1/k+m+n+1)} \quad (1.2) \end{aligned}$$

provided $\operatorname{Re} \sigma > -1$, m, n being integers.

2. Expanding $J_s \left\{ 2 (xt)^{k/2} \right\}$ and integrating term by term with the help of (1.1), we obtain

$$\begin{aligned} & \int_0^1 x^{k\sigma} (1-x^k)^{-n} Q_{mk}^{nk}(x) J_s \left\{ 2 (xt)^{k/2} \right\} dx \\ &= \frac{k^{2n-1} t^{sk/2} \Gamma(\sigma+s/2+1) \Gamma(\sigma+s/2+1/k)}{\Gamma(s+1) \Gamma(\sigma+s/2+1-m-n) \Gamma(\sigma+s/2+1/k+m-n+1)} \times \\ & {}_2F_3 \left[\begin{matrix} \sigma+s/2+1, \sigma+s/2+1/k \\ s+1, \sigma+s/2+1-m-n, \sigma+s/2+1/k+m-n+1 \end{matrix} ; -t^k \right], \quad (2.1) \end{aligned}$$

$\operatorname{Re} (k\sigma + \frac{ks}{2}) > -1$, s being an integer,

and with the help of (1.2), we have

$$\begin{aligned} & \int_0^1 x^{k\sigma} (1-x^k)^n Q_{mk}^{nk}(x) J_s \left\{ 2 (xt)^{k/2} \right\} dx \\ &= \frac{k^{2n-1} t^{sk/2} \Gamma(m+n+1) \Gamma(m+n+1/k) \Gamma(\sigma+s/2+1) \Gamma(\sigma+s/2+1/k)}{\Gamma(m-n+1) \Gamma(m-n+1/k) \Gamma(\sigma+s/2+1-m+n) \Gamma(\sigma+s/2+1/k+m-n+1)} \\ & {}_2F_3 \left[\begin{matrix} \sigma+s/2+1, \sigma+s/2+1/k \\ s+1, \sigma+s/2+1-m+n, \sigma+s/2+1/k+m-n+1 \end{matrix} ; -t^k \right], \quad (2.2) \end{aligned}$$

$\operatorname{Re} (k\sigma + sk/2) > -1$, s being an integer.

3. Since

$$\begin{aligned} & {}_2F_3 \left[\begin{matrix} \frac{1}{2}(\mu+\lambda+1), \frac{1}{2}(\mu+\lambda+2) \\ \mu+1, \lambda+1, \mu+\lambda+1 \end{matrix} ; -x^2 t^2 \right] \\ &= \Gamma(\mu+1) \Gamma(\lambda+1) (xt/2)^{-\lambda-\mu} J_\mu(xt) J_\lambda(xt). \end{aligned}$$

Replacing (xt) by $(xt)^{k/2}$, with the help of (1.1), we obtain

$$\int_0^1 x^{k\sigma - k\lambda/2 - k\mu/2} (1-x^k)^{-n} Q_{mk}^{nk}(x) J_\mu \left\{ (xt)^{k/2} \right\} J_\lambda \left\{ (xt)^{k/2} \right\} dx$$

$$= \frac{k^{2n-1} t^{k/2} (\lambda + \mu) \Gamma(\sigma+1) \Gamma(\sigma+1/k)}{2^{\lambda+\mu} \Gamma(\mu+1) \Gamma(\lambda+1) \Gamma(\sigma+1-m-n) \Gamma(\sigma+1/k+m-n+1)} \times$$

$${}_4F_5 \left[\begin{matrix} \frac{1}{2}(\mu+\lambda+1), \frac{1}{2}(\mu+\lambda+2), \sigma+1, \sigma+1/k \\ \mu+1, \lambda+1, \mu+\lambda+1, \sigma+1-m-n, \sigma+1/k+m-n+1 \end{matrix} ; -t^k \right], \quad (3.1)$$

$\operatorname{Re} k\sigma > -1$;

and with the help of (1.2), we have

$$\int_0^1 x^{k\sigma - k\lambda/2 - k\mu/2} (1-x^k)^n Q_{mk}^{nk}(x) J_\mu \left\{ (xt)^{k/2} \right\} J_\lambda \left\{ (xt)^{k/2} \right\} dx$$

$$= \frac{k^{2n-1} t^{k/2} (\lambda + \mu) \Gamma(m+n+1) \Gamma(m+n+1/k) \Gamma(\sigma+1) \Gamma(\sigma+1/k)}{2^{\lambda+\mu} \Gamma(\mu+1) \Gamma(\lambda+1) \Gamma(m-n+1) \Gamma(m-n+1/k) \Gamma(\sigma+1-m+n) \Gamma(\sigma+1/k+m-n+1)} \times$$

$${}_4F_5 \left[\begin{matrix} \frac{1}{2}(\mu+\lambda+1), \frac{1}{2}(\mu+\lambda+2), \sigma+1, \sigma+1/k \\ \mu+1, \lambda+1, \mu+\lambda+1, \sigma+1-m+n, \sigma+1/k+m-n+1 \end{matrix} ; -t^k \right], \quad (3.2)$$

$\operatorname{Re} k\sigma > -1$.

Next using the result of Meijer [5]

$${}_0F_3 \left[\begin{matrix} \frac{1}{2}+\mu, 1+\mu, 1+2\mu \\ \frac{1}{2}+\mu/2, 1+\mu/2, 1+\mu \end{matrix} ; x^2/6 \right] \frac{x^{2\mu}}{\{\Gamma(1-2\mu)\}^2 2^{4\mu}} = I_{2\mu}(\sqrt{x}) J_{2\mu}(\sqrt{x})$$

and proceeding as above, we obtain

$$\int_0^1 x^{k\sigma - \frac{k\mu}{2}} (1-x^k)^{-n} Q_{mk}^{nk}(x) I_\mu(x^{k/4}) J_\mu(x^{k/4}) dx$$

$$= \frac{k^{2n-1} \Gamma(\sigma+1) \Gamma(\sigma+1/k)}{\{\Gamma(1-\mu)\}^2 2^{2\mu} \Gamma(\sigma+1-m-n) \Gamma(\sigma+1/k+m-n+1)}$$

$${}_2F_5 \left[\begin{matrix} \sigma+1, \sigma+1/k \\ \frac{1}{2}+\mu/2, 1+\mu/2, 1+\mu, \sigma+1-m-n, \sigma+1/k+m-n+1 \end{matrix} ; -1/6 \right]; \quad (3.3)$$

and

$$\int_0^1 x^{k\sigma - k\mu/2} (1-x^k)^n Q_{mk}^{nk}(x) I_\mu(x^{k/4}) J_\mu(x^{k/4}) dx$$

$$= \frac{k^{2n-1} \Gamma(m+n-1) \Gamma(m+n+1/k) \Gamma(\sigma+1) \Gamma(\sigma+1/k)}{\{\Gamma(1-\mu)\}^2 2^{2\mu} \Gamma(m-n+1) \Gamma(m-n+1/k) \Gamma(\sigma+1-m+n) \Gamma(\sigma+1/k+m+n+1)} \times$$

$${}_2F_3 \left[\begin{matrix} \sigma+1, \sigma+1/k \\ \frac{1}{2}+\mu/2, 1+\mu/2, 1+\mu, \sigma+1-m+n, \sigma+1/k+m+n+1 \end{matrix} ; -1/6 \right], \quad (3.4)$$

$\operatorname{Re} k\sigma > -1$.

4. Consider the following relations (Watson [7], § 52.1, 5.23)

$$(\lambda t)^\lambda J_\mu(2\lambda t) = \sum_{\gamma=0}^{\infty} \frac{(-1)^\gamma \Gamma(\lambda+\mu+\gamma)}{\gamma! \Gamma(\mu+\gamma+1)} (\lambda)_\gamma (\lambda+\mu+2\gamma) J_{\mu+\lambda+2\gamma}(2\lambda t), \quad (4.1)$$

$$(\lambda t)^{\mu-\nu} J_\nu(2\lambda t) = \Gamma(\mu+1) \sum_{\gamma=0}^{\infty} \frac{(v-\mu)_\gamma (t/2)^\gamma}{\Gamma(v+\gamma+1)} J_{\mu+\gamma}(2\lambda t). \quad (4.2)$$

Replacing (λt) by $(\lambda t)^{k/2}$, multiplying the two sides by

$x^{k\sigma} (1-x^k)^{-n} Q_{mk}^{nk}(x)$ and integrating between the limits 0 and 1 with respect to x , we obtain

$${}_2F_3 \left[\begin{matrix} \sigma+\lambda/2+\mu/2+1, \sigma+\lambda/2+\mu/2+1/k \\ \mu+1, \sigma+\lambda/2+\mu/2+1-m-n, \sigma+\lambda/2+\mu/2+1/k+m-n+1 \end{matrix} ; -t^k \right]$$

$$= \sum_{\gamma=0}^{\infty} \frac{(-1)^\gamma \Gamma(\lambda+\mu+\gamma) (\lambda)_\gamma (\sigma+\lambda/2+\mu/2+1)_\gamma (\sigma+\lambda/2+\mu/2+1/k)_\gamma}{\gamma! (\mu+1)_\gamma \Gamma(\lambda+\mu+2\gamma) (\sigma+\lambda/2+\mu/2+1-m-n)_\gamma (\sigma+\lambda/2+\mu/2+1/k+m-n+1)_\gamma} \times$$

$${}_2F_3 \left[\begin{matrix} \sigma+\lambda/2+\mu/2+\gamma+1, \sigma+\lambda/2+\mu/2+\gamma+1/k \\ \lambda+\mu+2\gamma+1, \sigma+\lambda/2+\mu/2+\gamma+1-m-n, \sigma+\lambda/2+\mu/2+\gamma+1/k+m-n+1 \end{matrix} ; -t^k \right], \quad (4.3)$$

$$\operatorname{Re} \left(k\sigma + \frac{k\lambda}{2} + \frac{k\mu}{2} \right) > -1.$$

Similarly replacing λt by $(\lambda t)^{k/2}$, multiplying the two sides by $x^{k\sigma} (1-x^k)^{-n} Q_{mk}^{nk}(x)$ and integrating between the limits 0 and 1 with respect to x , we obtain

$${}_2F_3 \left[\begin{matrix} \sigma+\lambda/2+\mu/2+1, \sigma+\lambda/2+\mu/2+1/k \\ \mu+1, \sigma+\lambda/2+\mu/2+1-m-n, \sigma+\lambda/2+\mu/2+1/k+m-n+1 \end{matrix} ; -t^k \right]$$

$$\begin{aligned}
&= \sum_{\gamma=0}^{\infty} \frac{(-1)^{\gamma} \Gamma(\lambda + \mu + \gamma) (\lambda)_{\gamma} (\sigma + \lambda/2 + \mu/2 + 1)_{\gamma} (\sigma + \lambda/2 + \mu/2 + 1/k)_{\gamma} t^{k\gamma}}{\gamma! (\mu + 1)_{\gamma} \Gamma(\lambda + \mu + 2\gamma) (\sigma + \lambda/2 + \mu/2 + 1 - m + n)_{\gamma} (\sigma + \lambda/2 + \mu/2 + 1/k + m + n + 1)_{\gamma}} \\
&{}_2F_3 \left[\begin{matrix} \sigma + \lambda/2 + \mu/2 + \gamma + 1, \sigma + \lambda/2 + \mu/2 + \gamma + 1/k \\ \lambda + \mu + 2\gamma + 1, \sigma + \lambda/2 + \mu/2 + \gamma + 1 - m + n, \sigma + \lambda/2 + \mu/2 + \gamma + 1/k + m + n + 1 \end{matrix} ; -t^k \right],
\end{aligned} \tag{4.4}$$

$$\operatorname{Re} \left(k\sigma + \frac{k\lambda}{2} + \frac{k\mu}{2} \right) > -1.$$

Similarly the results obtained by using (4.2) are

$$\begin{aligned}
&{}_2F_3 \left[\begin{matrix} \sigma + \mu/2 + 1, \sigma + \mu/2 + 1/k \\ \nu + 1, \sigma + \mu/2 + 1 - m - n, \sigma + \mu/2 + 1/k + m - n + 1 \end{matrix} ; -t^k \right] \\
&= \sum_{\gamma=0}^{\infty} \frac{(\nu - \mu)_{\gamma} \Gamma(\sigma + \mu/2 + \gamma/2 + 1) \Gamma(\sigma + \mu/2 + \gamma/2 + 1/k)}{(\nu + 1)_{\gamma} (\mu + 1)_{\gamma} \Gamma(\sigma + \mu/2 + 1) \Gamma(\sigma + \mu/2 + 1/k)} \times \\
&\quad \frac{\Gamma(\sigma + \mu/2 + 1 - m - n) \Gamma(\sigma + \mu/2 + 1/k + m - n + 1)}{\Gamma(\sigma + \mu/2 + \gamma/2 + 1 - m - n) \Gamma(\sigma + \mu/2 + \gamma/2 + 1/k + m - n + 1)} \left(\frac{t^k}{2} \right)^{\gamma} \times \\
&{}_2F_3 \left[\begin{matrix} \sigma + \mu/2 + \gamma/2 + 1, \sigma + \mu/2 + \gamma/2 + 1/k \\ \mu + \gamma + 1, \sigma + \mu/2 + \gamma/2 + 1 - m - n, \sigma + \mu/2 + \gamma/2 + 1/k + m - n + 1 \end{matrix} ; -t^k \right],
\end{aligned} \tag{4.5}$$

$$\operatorname{Re} \left(k\sigma + \frac{k\mu}{2} \right) > -1; \text{ and}$$

$$\begin{aligned}
&{}_2F_3 \left[\begin{matrix} \sigma + \mu/2 + 1, \sigma + \mu/2 + 1/k \\ \nu + 1, \sigma + \mu/2 + 1 - m + n, \sigma + \mu/2 + 1/k + m + n + 1 \end{matrix} ; -t^k \right] \\
&= \sum_{\gamma=0}^{\infty} \frac{(\nu - \mu)_{\gamma} \Gamma(\sigma + \mu/2 + \gamma/2 + 1) \Gamma(\sigma + \mu/2 + \gamma/2 + 1/k)}{(\nu + 1)_{\gamma} (\mu + 1)_{\gamma} \Gamma(\sigma + \mu/2 + 1) \Gamma(\sigma + \mu/2 + 1/k)} \times \\
&\quad \frac{\Gamma(\sigma + \mu/2 + 1 - m + n) \Gamma(\sigma + \mu/2 + 1/k + m + n + 1)}{\Gamma(\sigma + \gamma/2 + \gamma/2 + 1 - m + n) \Gamma(\sigma + \mu/2 + \gamma/2 + 1/k + m + n + 1)} \left(\frac{t^k}{2} \right)^{\gamma} \times \\
&{}_2F_3 \left[\begin{matrix} \sigma + \mu/2 + \gamma/2 + 1, \sigma + \mu/2 + \gamma/2 + 1/k \\ \mu + \gamma + 1, \sigma + \mu/2 + \gamma/2 + 1 - m + n, \sigma + \mu/2 + \gamma/2 + 1/k + m + n + 1 \end{matrix} ; -t^k \right],
\end{aligned} \tag{4.6}$$

$$\operatorname{Re} \left(k\sigma + \frac{k\mu}{2} \right) > -1.$$

5. We now obtain the results which are most general. Following the same method, we have

$$\int_0^1 x^{k\sigma} (1-x^k)^{-n} Q_{mk}^{nk}(x) {}_pF_q \left[\begin{matrix} \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q \end{matrix} ; \pm (xt)^k \right] dx$$

$$= \frac{k^{2n-1} \Gamma(\sigma+1) \Gamma(\sigma+1/k)}{\Gamma(\sigma+1-m-n) \Gamma(\sigma+1/k+m-n+1)} \times$$

$${}_{p+2}F_{q+2} \left[\begin{matrix} \alpha_1, \dots, \alpha_p, \sigma+1, \sigma+1/k \\ \beta_1, \dots, \beta_q, \sigma+1-m-n, \sigma+1/k+m-n+1 \end{matrix} ; \pm (t)^k \right], \quad (5.1)$$

$\text{Re } k\sigma > -1$; and

$$\int_0^1 x^{k\sigma} (1-x^k)^n Q_{mk}^{nk}(x) {}_pF_q \left[\begin{matrix} \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q \end{matrix} ; \pm (xt)^k \right] dx$$

$$= \frac{k^{2n-1} \Gamma(m+n+1) \Gamma(m+n+1/k) \Gamma(\sigma+1) \Gamma(\sigma+1/k)}{\Gamma(m-n+1) \Gamma(m-n+1/k) \Gamma(\sigma+1-m+n) \Gamma(\sigma+1/k+m+n+1)} \times$$

$${}_{p+2}F_{q+2} \left[\begin{matrix} \alpha_1, \dots, \alpha_p, \sigma+1, \sigma+1/k \\ \beta_1, \dots, \beta_q, \sigma+1-m+n, \sigma+1/k+m+n+1 \end{matrix} ; \pm (t)^k \right], \quad (5.2)$$

$\text{Re } k\sigma > -1$.

For $k=2$ all the above results with certain restrictions correspond to the results of Bhonsle [2].

6. A generalization of Hermite Polynomials given by Chatterjee [3] is as follows :

$$H_{km}(x) = (-1)^m k^{2m} \frac{\Gamma(m+1/k)}{\Gamma(1/k)} {}_1F_1 \left(-m; 1/k; x^k \right)$$

$$H_{km+1}(x) = (-1)^m k^{2m+1} \frac{\Gamma(m+1/k+1)}{\Gamma(1/k+1)} {}_1F_1 \left(-m; 1/k+1; x^k \right).$$

Using these results, proceeding as above, we obtain

$$\int_0^1 x^{k\sigma} (1-x^k)^{-n} Q_{mn}^{nk}(x) H_{ks}(x) dx$$

$$= (-1)^s \frac{k^{2s+2n-1} \Gamma(s+1/k) \Gamma(\sigma+1) \Gamma(\sigma+1/k)}{\Gamma(1/k) \Gamma(\sigma+1-m-n) \Gamma(\sigma+1/k+m-n+1)} \times$$

$${}_3F_3 \left[\begin{matrix} -s, \sigma+1, \sigma+1/k \\ 1/k, \sigma+1-m-n, \sigma+1/k+m-n+1 \end{matrix} ; 1 \right], \quad (6.1)$$

$\operatorname{Re} k\sigma > -1$;

and

$$\begin{aligned} & \int_0^1 x^{k\sigma} (1-x^k)^n Q_{mk}^{nk}(x) H_{ks}(x) dx \\ &= \frac{(-1)^s k^{2s+2n-1} \Gamma(s+1/k) \Gamma(m+n+1) \Gamma(m+n+1/k) \Gamma(\sigma+1) \Gamma(\sigma+1/k)}{\Gamma(1/k) \Gamma(m-n+1) \Gamma(m-n+1/k) \Gamma(\sigma+1-m+n) \Gamma(\sigma+1/k+m+n+1)} \\ & \quad {}_3F_3 \left[\begin{matrix} -s, \sigma+1, \sigma+1/k \\ 1/k, \sigma+1-m+n, \sigma+1/k+m+n+1 \end{matrix} ; 1 \right], \end{aligned} \quad (6.2)$$

$\operatorname{Re} k\sigma > -1$.

Similar results can be obtained for $H_{ks+1}(x)$.

Next consider the following relation (Bailey [1])

$${}_3F_3 \left[\alpha, \rho-\alpha; \rho, \frac{\rho}{2}, \frac{\rho}{2}+1; \frac{x^2}{4} \right] = e^{-x} {}_1F_1(\alpha; \rho; x) {}_1F_1(\rho-\alpha; \rho; x).$$

Replacing x by $(x)^{k/2}$, using (5.1) and (5.2), we obtain

$$\begin{aligned} & \int_0^1 x^{k\sigma} (1-x^k)^n Q_{mk}^{nk}(x) e^{-(x^{k/2})} {}_1F_1(\alpha; \rho; x^{k/2}) {}_1F_1(\rho-\alpha; \rho; x^{k/2}) dx \\ &= \frac{k^{2n-1} \Gamma(\sigma+1) \Gamma(\sigma+1/k)}{\Gamma(\sigma+1-m-n) \Gamma(\sigma+1/k+m-n+1)} \\ & \quad {}_4F_5 \left[\begin{matrix} \alpha, \rho-\alpha, \sigma+1, \sigma+1/k \\ \rho, \rho/2, \rho/2+1, \sigma+1-m-n, \sigma+1/k+m-n+1 \end{matrix} ; \frac{1}{4} \right], \end{aligned} \quad (6.3)$$

$\operatorname{Re} k\sigma > -1$;

and

$$\begin{aligned} & \int_0^1 x^{k\sigma} (1-x^k)^n Q_{mk}^{nk}(x) e^{-(x^{k/2})} {}_1F_1(\alpha; \rho; x^{k/2}) {}_1F_1(\rho-\alpha; \rho; x^{k/2}) dx \\ &= \frac{k^{2n-1} \Gamma(\sigma+m+1) \Gamma(\sigma+m+1/k) \Gamma(\sigma+1) \Gamma(\sigma+1/k)}{\Gamma(m-n+1) \Gamma(m-n+1/k) \Gamma(\sigma+1-m+n) \Gamma(\sigma+1/k+m+n+1)} \times \\ & \quad {}_4F_5 \left[\begin{matrix} \alpha, \rho-\alpha, \sigma+1, \sigma+1/k \\ \rho, \rho/2, \rho/2+1, \sigma+1-m+n, \sigma+1/k+m+n+1 \end{matrix} ; \frac{1}{4} \right], \end{aligned} \quad (6.4)$$

$\operatorname{Re} k\sigma > -1$.

Since

$$L_n^\alpha(x) = \frac{\Gamma(\alpha+1+n)}{n! \Gamma(\alpha+1)} {}_1F_1(-n; \alpha+1; x),$$

$$M_{k,m}(x) = x^{m+\frac{1}{2}} e^{-x/2} {}_1F_1(m+\frac{1}{2}-k; 2m+1; x),$$

using these relations we can easily deduce from (6.3) and (6.4) the following special cases

$$\begin{aligned} \int_0^1 x^{k\sigma} (1-x^k)^{-n} Q_{mk}^{nk}(x) e^{-x^{k/2}} L_l^\alpha(x^{k/2}) M_{-l-\frac{1}{2}}(\alpha+1, \alpha/2)(x^{k/2}) dx \\ = \frac{k^{2n-1}}{l!} \frac{\Gamma(\alpha+1+l) \Gamma(\sigma+\alpha/4+5/4) \Gamma(\sigma+\alpha/4+1/4+1/k)}{\Gamma(\alpha+1) \Gamma(\sigma+\alpha/4+5/4-m-n) \Gamma(\sigma+\alpha/4+5/4+1/k+m-n)} \times \\ {}_4F_5 \left[\begin{matrix} -l, \alpha+l+1, \sigma+\alpha/4+5/4, \sigma+\alpha/4+1/4+1/k \\ \alpha+1, \alpha/2+1/2, \alpha/2+1, \sigma+\alpha/4+5/4-m-n, \sigma+\alpha/4+5/4+m-n \end{matrix} ; \frac{1}{k} \right], \end{aligned}$$

$$\operatorname{Re} \left\{ k\sigma + \frac{k(\alpha+1)}{4} \right\} > -1;$$

and

$$\begin{aligned} \int_0^1 x^{k\sigma} (1-x^k)^n Q_{mk}^{nk}(x) e^{-x^{k/2}} L_l^\alpha(x^{k/2}) M_{-l-\frac{1}{2}}(\alpha+1, \alpha/2)(x^{k/2}) dx \\ = \frac{k^{2n-1}}{l!} \frac{\Gamma(\alpha+1+l) \Gamma(m+n+1) \Gamma(m+n+1/k) \Gamma(\sigma+\alpha/4+5/4) \Gamma(\sigma+\alpha/4+1/4+1/k)}{\Gamma(\alpha+1) \Gamma(m-n+1) \Gamma(m-n+1/k) \Gamma(\sigma+\alpha/4+5/4-m+n) \Gamma(\sigma+\alpha/4+5/4+1/k+m+n)}, \\ {}_4F_5 \left[\begin{matrix} -l, \alpha+l+1, \sigma+\alpha/4+5/4, \sigma+\alpha/4+1/4+1/k \\ \alpha+1, \alpha/2+1/2, \alpha/2+1, \sigma+\alpha/4+5/4-m+n, \sigma+\alpha/4+5/4+1/k+m+n \end{matrix} ; \frac{1}{k} \right], \end{aligned}$$

$$\operatorname{Re} \left\{ k\sigma + \frac{k(\alpha+1)}{4} \right\} > -1.$$

I am thankful to Dr. K. N. Srivastava for his help and guidance in the preparation of this note.

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IODINE MONOBROMIDE AS REDOX REAGENT DETERMINATION OF THIOUREA AND ITS ORGANIC DERIVATIVES

By

BALWANT SINGH, BALBIR CHAND VERMA and M. S. SARAN

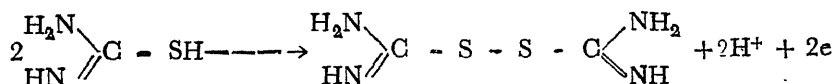
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ABSTRACT

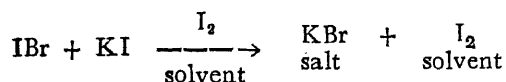
Iodine monobromide (in glacial acetic acid) solution is used as a redox reagent for the determination of thiourea and some of its alkyl and aryl derivatives both by visual and potentiometric titration methods.

The thioureas are oxidised to disulphides with a single electron change.



This direct method is simple, accurate and instantaneous.

Iodine monobromide behaves as an acid in iodine solution :



According to Schulek and Burger¹, iodine monobromide acts partly as an iodinating and partly as a brominating agent in preparative organic chemistry. Militzer² has found that in carbon disulphide or acetic acid, iodine monobromide functions as a brominating agent. Bennet and Sharpe³ have shown that in a solution of phenol and iodine monobromide in nitrobenzene, free bromine and iodine cations compete for the phenol. Since bromination of phenol is very fast as compared to its iodination, the resulting reaction is iodine catalysed bromination and iodine may be recovered at the end of the reaction (Lambourne and Robertson⁴).

Singh and Singh⁵ have used iodine monobromide (in anhydrous glacial acetic acid) solution for the determination of phenol, hydroquinone, resorcinol, pyrogallol, phloroglucinol, thymol, α - and β -naphthol, o- and p-cresol, o- and p-nitrophenol, aniline, o-, m- and p-toluidine, α -naphthyl amine, acetanilide, salicylic acid, anthranilic acid, p-aminobenzoic acid and naphthalene.

Quantitative estimation of thiourea or its organic derivatives with iodine monobromide has not been studied so far. Singh et al.⁶ have used iodine monochloride and iodine trichloride as redox reagents for the determination of thiourea and some of its alkyl and aryl derivatives. In the present paper an attempt has been made to use iodine monobromide as a redox reagent for the determination of thioureas by oxidising them to their corresponding disulphides. This direct method is simple, accurate and instantaneous.

EXPERIMENTAL

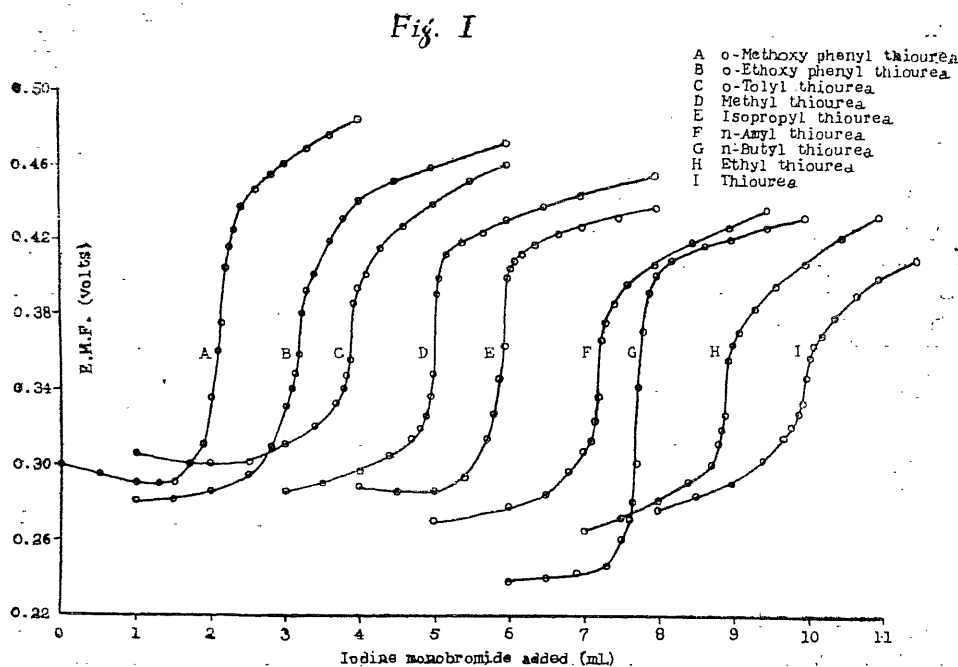
Iodine monobromide was prepared by warming a mixture of 0.55 gm-molecules of bromine and 0.50 gm-molecules of iodine on a water bath till the

iodine completely dissolved. The solid compound which separated on cooling was purified by repeated melting and cooling till its melting point was 42.0°C.

For preparing N/40 solution, 3 g. of the compound was dissolved in glacial acetic acid and the volume of the solution made to one litre with the acid. The solution was then standardised by an iodometric method and kept in the dark.

A known weight (5 to 40 mg.) of each compound was taken in a titration flask. In case of thiourea and its alkyl derivatives, sufficient water and enough of sulphuric acid to keep normality of the solution at 2N were added. Each aryl derivative of thiourea was dissolved in 18N sulphuric acid (10 to 12 ml.) and sufficient water (50 to 60 ml.) was added to keep normality of the solution at 3N. The solution in each case was cooled to room temperature and titrated with standard (0.025N) iodine monobromide (in glacial acetic acid) solution. End-point in each titration was detected visually as well as potentiometrically. In visual titrations amylose was used as an indicator and the solution acquired blue colour at the end point (Intense blue colour was not obtained due to the presence of acetate ions). In case of thiourea and its alkyl derivatives, chloroform was also used as an indicator which turned pink at the end point.

In potentiometric method a platinum wire immersed in the solution to be titrated was used as an oxidation-reduction electrode, and this was coupled with a saturated calomel electrode. The solution was kept stirred by a mechanical stirrer during the titration. Progress of the reaction was studied with Mullard potentiometer. At the equivalence point there was a sharp jump in potential in each titration. A series of potentiometric titrations was performed with different amounts of each compound. The potentiometric titrations, one for each compound, are represented by curves A to I.



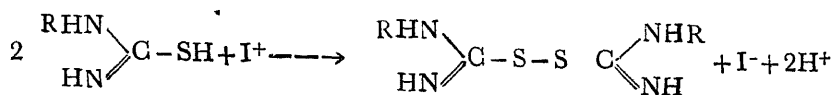
From the volume of standard (N/40 iodine monobromide (in glacial acetic acid) solution used corresponding to the end point in each titration, the amount of thiourea and each of its organic derivatives is calculated. Some typical results are given in Table I.

TABLE I

Titration of thiourea and its organic derivatives with standard (N/40) iodine monobromide solution.

Compound	Visual Method		Potentiometric Method	
	taken g	found g	taken g	found g
NH ₂ .CS.NH ₂	0.0095	0.0095	0.0133	0.0133
	0.0190	0.0190	0.0304	0.0303
CH ₃ NH.CS.NH ₂	0.0102	0.0102	0.0113	0.0113
	0.0320	0.0318	0.0360	0.0359
CH ₃ CH ₂ .NH.CS.NH ₂	0.0165	0.0165	0.0104	0.0104
	0.0390	0.0388	0.0417	0.0415
(CH ₃) ₂ CH.NH.CS.NH ₂	0.0114	0.0114	0.0176	0.0176
	0.0395	0.0393	0.0393	0.0392
<i>n</i> -CH ₃ .CH ₂ .CH ₂ .CH ₂ .NH.CS.NH ₂	0.0103	0.0103	0.0115	0.0115
	0.0349	0.0347	0.0383	0.0384
<i>n</i> -CH ₃ .CH ₂ .CH ₂ .CH ₂ .CH ₂ .NH.CS.NH ₂	0.0114	0.0114	0.0157	0.0157
	0.0404	0.0402	0.0400	0.0401
<i>o</i> -(CH ₃)C ₆ H ₄ .NH.CS.NH ₂	0.0108	0.0108	0.0162	0.0163
	0.0352	0.0351	0.0332	0.0332
<i>o</i> -(OCH ₃)C ₆ H ₄ .NH.CS.NH ₂	0.0110	0.0110	0.0167	0.0167
	0.0355	0.0353	0.0320	0.0320
<i>o</i> -(OC ₂ H ₅)C ₆ H ₄ .NH.CS.NH ₂	0.0105	0.0105	0.0158	0.0158
	0.0322	0.0321	0.0396	0.0394

The results recorded above show that the listed compounds can be determined by titration with standard (N/40) iodine monobromide (in glacial acetic acid) solution. Thiourea and its organic derivatives are oxidised by iodine monobromide to their corresponding disulphides:



(R = alkyl or aryl group).

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GENERALISED HANKEL TRANSFORMS AND SELF-RECIPROCAL FUNCTION*

By

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ABSTRACT

Roop Narain (1) has given a generalisation of the well known Hankel transform in the form

$$g(x) = \left(\frac{1}{2}\right)^{\nu} \int_0^{\infty} (xy)^{\nu + \frac{1}{2}} \chi_{\nu, k, m} \left(\frac{1}{2} x^2 y^2\right) f(y) dy$$

In this paper, some properties in the form of theorems have been given. These properties are more general in nature than the known ones.

1. Roop Narain¹ (1956-57) has given a generalisation of the well-known Hankel Transform, namely

$$f(x) = \int_0^{\infty} \sqrt{xy} J_{\nu}(xy) g(y) dy, \quad (1.1)$$

by means of an integral equation

$$f(x) = \left(\frac{1}{2}\right)^{\nu} \int_0^{\infty} (xy)^{\nu + \frac{1}{2}} \chi_{\nu, k, m} \left(\frac{1}{4} x^2 y^2\right) g(y) dy, \quad (1.2)$$

where $\chi_{\nu, k, m}(x)$ is a function, which may be expressed in its simplest form by

Meijer G-function as

$$\chi_{\nu, k, m}(x) = x^{-\nu} G_{2, 4}^{2, 1} \left(x \left| \begin{matrix} k+m-\frac{1}{2}, \nu-k+m+\frac{1}{2} \\ \nu, \nu+2m, 0, -2m \end{matrix} \right. \right), \quad (1.3)$$

provided $R(s) \geq s_0 > 0$, $R(\nu + 1 + 2m \pm 2m) > 0$ and $2m$ is not an integer or zero; and the integrals involved are absolutely convergent.

In particular, when $k+m = \frac{1}{2}$; (1.2) reduces to (1.1). He has also established the reciprocal relation, namely

$$g(x) = 1/2^{\nu} \int_0^{\infty} (xy)^{\nu + \frac{1}{2}} \chi_{\nu, k, m} (1/4 x^2 y^2) f(y) dy, \quad (1.4)$$

*Read at the Mathematical Conference, 1961.

provided the $\chi_{\nu,k,m}$ -transforms of $|f(x)|$ and $|g(x)|$ exist, $R(s) \geq R(s_0) > 0$, $R(\nu + 1 + 2m \pm 2m)$, and $2m$ is not an integer. He defined $f(x)$ given by (1.2) as the $\chi_{\nu,k,m}$ -transform of $g(x)$. If $f(x) = g(x)$, so that $f(x)$ is its own $\chi_{\nu,k,m}$ -transform $f(x)$ is said to be self-reciprocal in the $\chi_{\nu,k,m}$ -transform and is written as $R_{\nu}(k, m)$. He has obtained the necessary and sufficient condition for a function to be self-reciprocal in this transform. He has given some rules connecting different classes of self-reciprocal functions. He has assumed that $f(x)$ is integrable in $(0, \infty)$ and the integrals involved exist.

The object of this paper is to develop the theory of this generalised transform. I have deduced some theorems which are exactly analogous to the corresponding results in the ordinary Hankel transform.

2. *Theorem 1.* If $f(x)$ and $g(x)$ are $\chi_{\nu,k,m}$ -transforms of each other, then

$$\int_0^{\infty} \frac{f(x)}{x^p} dx = \frac{2^{1/2-p} \Gamma_x(1/2\nu - 1/2p + 3/4 + m \pm m) \Gamma(\nu/2 + p/2 + 3/4 + m - k)}{\Gamma(\nu/2 - p/2 + 5/4 + m - k) \Gamma_x(\nu/2 + p/2 + 1/4 + m \pm m)} \int_0^{\infty} \frac{g(x)}{x^{2-p}} dx, \quad (2.1)$$

provided the integrals converge, and $2m$ is not an integer.

Proof: - We know that $f(x) = 1/2\nu \int_0^{\infty} (xy)^{\nu+1/2} \chi_{\nu,k,m}(1/4 x^2 y^2) g(y) dy$

Therefore

$$\int_0^{\infty} \frac{f(x)}{x^p} dx = \int_0^{\infty} \frac{dx}{x^p} \cdot 1/2\nu \int_0^{\infty} (xy)^{\nu+1/2} \chi_{\nu,k,m}(1/4 x^2 y^2) g(y) dy. \quad (2.2)$$

Changing the order of integration, if permissible, we get

$$\int_0^{\infty} \frac{f(x)}{x^p} dx = 1/2\nu \int_0^{\infty} y^{\nu+1/2} g(y) dy \int_0^{\infty} y^{\nu-p+1/2} \chi_{\nu,k,m}(1/4 x^2 y^2) dx. \quad (2.3)$$

Evaluating the inner integral by a formula given by Roop Narain¹ which is of the form:

$$\int_0^{\infty} y^{\mu-1} \chi_{\nu,k,m}(1/4 x^2 y^2) dy = \frac{2^{\mu-1} \Gamma_x(\mu/2 + m \pm m) \Gamma(\nu - \mu/2 + 3/2 + m - k)}{x^{\mu} \Gamma(\mu/2 + 1/2 + m - k) \Gamma_x(\nu - \mu/2 + 1 + m \pm m)} \quad (2.3A)$$

provided $R(\nu + 1 + 4m) > R(\mu/2 + 2m) > 0$, $R(\nu + 3/2 + m - k - \mu/2) > 0$,

$R(1/2 + k - \mu/2 - m) > 0$, $R(\mu + \frac{1}{2}) > R(\mu) > 0$, and $2m$ is not an integer or zero,

*For the sake of brevity $\Gamma_x(\alpha \pm \beta)$ has been employed to denote $\Gamma(\alpha + \beta) \Gamma(\alpha - \beta)$.

We get

$$\int_0^{\infty} \frac{f(x)}{x^p} dx = \frac{2^{1/2} \Gamma(\nu/2 + 3/4 - p/2 + m \pm m) \Gamma(\nu/2 + p/2 + 3/4 + m - k)}{\Gamma(\nu/2 - p/2 + 5/4 + m - k) \Gamma(\nu/2 + p/2 + 1/4 + m \pm m)} \int_0^{\infty} \frac{g(x)}{x^{1-p}} dx.$$

The change in the order of integration is permissible as the integrals involved are absolutely convergent.

Particular Cases:—

(i) When $p = \frac{1}{2}$, we get

$$\int_0^{\infty} \frac{f(x)}{\sqrt{x}} dx = \int_0^{\infty} \frac{g(x)}{x^{1-p}} dx, \quad (2.4)$$

(ii) When $k + m = \frac{1}{2}$, we have

$$\int_0^{\infty} \frac{f(x)}{x^p} dx = \frac{2^{\frac{1}{2}-p} \Gamma(\nu/2 + 3/4 - p/2)}{\Gamma(\nu/2 + 1/4 + p/2)} \int_0^{\infty} \frac{g(x)}{x^{1-p}} dx, \quad (2.5)$$

This theorem is a particular case of Parseval formula for Hankel transforms, which is,

$$\int_0^{\infty} f(xy) g(y) dy = \int_0^{\infty} f(y) g(xy) dy. \quad (2.6)$$

(iii) When $p = \frac{1}{2}$ in (2.5), we get

$$\int_0^{\infty} \frac{f(x)}{\sqrt{x}} dx = \int_0^{\infty} \frac{g(x)}{\sqrt{x}} dx.$$

Here $f(x)$ and $g(x)$ are ordinary Hankel-transforms of each other.

Ex. 1. Let us take $f(x) = x^{\nu+2m-\frac{1}{2}} e^{-\frac{1}{4}x^2} W_{k,m}(\frac{1}{2}x^2)$.

Then the corresponding $\chi_{\nu,k,m}$ -transform of $f(x)$ is $g(x) = x^{\nu+2m} e^{-\frac{1}{4}x^2} \times W_{k,m}(\frac{1}{2}x^2)$, so that $f(x) = g(x)$. Hence $f(x)$ is a self-reciprocal function in $\chi_{\nu,k,m}$ -transform.

By the theorem, we have

$$\begin{aligned} & \int_0^{\infty} \frac{x^{\nu+2m-\frac{1}{2}} e^{-\frac{1}{4}x^2} W_{k,m}(\frac{1}{2}x^2)}{x^p} dx \\ &= \frac{2^{\frac{1}{2}-p} \Gamma(\nu/2 - p/2 + 3/4) \Gamma(\nu/2 - p/2 + 3/4 + 2m) \Gamma(\nu/2 + p/2 + 3/4 + m - k)}{\Gamma(\nu/2 + p/2 + 1/4 + m \pm m) \Gamma(\nu/2 - p/2 + 5/4 + m - k)} \\ & \quad \int_0^{\infty} \frac{x^{\nu+2m-\frac{1}{2}} e^{-\frac{1}{4}x^2} W_{k,m}(x^2/2)}{x^{1-p}} dx. \end{aligned}$$

Evaluating the integrals by a well-known formula given by Goldstein⁴,

$$\int_0^{\infty} x^{2l-1} e^{-\frac{1}{2}x^2} W_{k,m}(\frac{1}{2}x^2) dx = \frac{\Gamma(l+m+\frac{1}{2}) \Gamma(l-m+\frac{1}{2})}{2^{1-l} \Gamma(l-k+1)},$$

where $l \pm m + \frac{1}{2} > 0$,
we get

$$\frac{\Gamma x(v/2+3/4-p/2 \pm m)}{2^{1-v/2-p/2-m+1/4} \Gamma(v/2+p/2+3/4+m-k)} = \frac{\Gamma x(v/2+3/4-p/2+m \pm m)}{2^{1-v/2-m-p/2+1/4} \Gamma(v/2+p/2+3/4+m-k)}$$

Thus the theorem is established.

Ex. 2. Let us take $f(x) = x^{v+2m+1/2} k_{2m}(ax)$. The corresponding $\chi_{v,k,m}$ -

transform of $f(x)$ is $g(x) = x^{v+2m} a^{2m} \Gamma(v+1+2m) x^v (a^2+x^2)^{-v-1-2m}$.
By the theorem we have

$$\int_0^{\infty} \frac{x^{v+2m+1/2}}{x^p} k_{2m}(ax) dx = \frac{2^{1/2-p} \Gamma x(v/2-p/2+3/4+m \pm m) \Gamma(v/2+p/2+3/4+m-k)}{\Gamma(v/2+p/2+5/4+m-k) \Gamma x(v/2+p/2+1/4+m \pm m)} \\ x \Gamma(v+1+2m) 2^{v+2m} a^{2m} \int_0^{\infty} \frac{x^v (a^2+x^2)^{-v-1-2m}}{x^{1-p}} dx.$$

Evaluating the integral on the left hand side by a well-known formula⁷:

$$\int_0^{\infty} x^{m-1} k_n(ax) dx = \frac{2^{m-2} \Gamma(m/2+n/2) \Gamma(m/2-n/2)}{a^m},$$

we get

$$\int_0^{\infty} x^{v+2m-p+1/2} k_{2m}(ax) dx = \frac{2^{v-p+2m-1/2}}{a^{v-p+2m+3/2}} \Gamma(2m+v/2-p/2+3/4) \Gamma(v/2-p/2+3/4)$$

Evaluating the integral on the right-hand side, we get

$$\text{R. H. S.} = \frac{2^{v-p+2m-1/2}}{a^{v-p+2m+3/2}} \Gamma x(v/2-p/2+3/4+m \pm m)$$

i. e. L. H. S. = R. H. S. Thus the theorem is established.

3. *Theorem 2.* If $f_1(x)$ and $f_2(x)$ are $\chi_{\nu, k, m}$ -transform of $g_1(x)$ and $g_2(x)$

respectively, then

$$\int_0^{\infty} f_1(x) g_2(x) dx = \int_0^{\infty} f_2(x) g_1(x) dx, \quad (3.1)$$

provided the integrals exist.

This may be called the Parseval theorem for this generalised Hankel-transform.

Proof:—We know that

$$f_1(x) = 1/2\nu \int_0^{\infty} (xy)^{\nu+\frac{1}{2}} \chi_{\nu, k, m} (1/4 x^2 y^2) g_1(y) dy, \quad (3.2)$$

$$\text{and } f_2(x) = 1/2\nu \int_0^{\infty} (xy)^{\nu+\frac{1}{2}} \chi_{\nu, k, m} (\frac{1}{4} x^2 y^2) g_2(y) dy, \quad (3.3)$$

$$\text{Therefore } \int_0^{\infty} f_1(x) g_2(x) dx = \int_0^{\infty} g_2(x) dx \cdot 1/2\nu \int_0^{\infty} (xy)^{\nu+\frac{1}{2}} \chi_{\nu, k, m} (\frac{1}{4} x^2 y^2) g_1(y) dy \quad (3.4)$$

Changing the order of integration, if permissible, we get

$$\begin{aligned} \int_0^{\infty} f_1(x) g_2(x) dx &= 1/2\nu \int_0^{\infty} g_1(y) dy \cdot \int_0^{\infty} (xy)^{\nu+\frac{1}{2}} \chi_{\nu, k, m} (\frac{1}{4} x^2 y^2) g_2(x) dx \\ &= \int_0^{\infty} g_1(x) \cdot f_2(x) dx; \end{aligned}$$

which proves the theorem.

Assuming that the integrals (3.2) and (3.3) are absolutely convergent, the change in the order of integration will be justifiable:

Ex. 1. Let $f_1(x) = x^{\mu-\frac{1}{2}}$

$$g_1(x) = \frac{2^{\mu} \Gamma x (\nu/2 + \mu/2 + 1/2 + m \pm m) \Gamma (\nu/2 - \mu/2 + 1 + m - k)}{x^{\mu+\frac{1}{2}} \Gamma (\mu/2 + \nu/2 + 1 + m - k) \Gamma x (\nu/2 + \mu/2 + 1/2 + m \pm m)}$$

$$\text{and } f_2(x) = x^{\nu+2m+\frac{1}{2}} k_{2m}(ax)$$

$$g_2(x) = 2^{\nu+2m} a^{2m} x^{\nu+\frac{1}{2}} \Gamma (\nu+1+2m) (a^2+x^2)^{-\nu-1-2m}$$

By the theorem, we have

$$\int_0^{\infty} x^{\mu-\frac{1}{2}} 2^{\nu+2m} a^{2m} \Gamma(\nu+1+2m) x^{\nu+\frac{1}{2}} (a^2+x^2)^{-\nu-1-2m} dx \\ = \frac{2^{\mu} \Gamma x(\mu/2+\nu/2+1/2+m \pm m) \Gamma(\nu/2-\mu/2+1+m-k)}{\Gamma(\mu/2+\nu/2+1+m-k) \Gamma x(\nu/2-\mu/2+1+m \pm m)}$$

The integral on the Right Hand side is

$$\frac{2^{\nu+\mu+2m-1} \Gamma x(\mu/2+\nu/2+\frac{1}{2}+m \pm m) \Gamma x(\nu/2-\mu/2+1+m-k) \Gamma x(\nu/2-\mu/2+\frac{1}{2}+m \pm m)}{\Gamma(\mu/2+\nu/2+1+m-k) \Gamma x(\nu/2+1/2-m/2+m \pm m) a^{\nu-\mu+2m-1}}$$

The integral on the left-hand side exists and hence will be equal to the right hand side.

4. *Theorem 3.* If $f(x)$ and $g(x)$ are $\chi_{\nu, k, m}$ - transforms of each other and

$$g(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \phi(s) x^{-s} ds,$$

where for $s = \sigma + it$, $\phi(s) = O(-k |t|)$, $k > 0$; and $|t| \rightarrow \infty$ and σ and is a value of c in the strip $0 < c < 1$, then

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s-\frac{1}{2}} \Gamma x(\nu/2+s/2+1/4+m \pm m) \Gamma(\nu/2-s/2+7/4+m-k)}{\Gamma(\nu/2+3/4+s/2+m-k) \Gamma x(\nu/2+s/2+3/4+m \pm m)} \times \\ \times \phi(1-s) x^{-s} ds;$$

provided $R(\nu+\frac{1}{2}) > R(\nu+\frac{1}{2}-s)$ and $2m$ is not an integer,

Proof:—We know that

$$f(x) = 1/2\nu \int_0^{\infty} (xy)^{\nu+\frac{1}{2}} \chi_{\nu, k, m}(\frac{1}{2} x^2 y^2) g(y) dy.$$

Substituting the value of $g(y)$ from (4.1), we get

$$f(x) = \frac{1}{2\nu} \int_0^{\infty} (xy)^{\nu+\frac{1}{2}} \chi_{\nu, k, m}(\frac{1}{2} x^2 y^2) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \phi(s) x^{-s} ds.$$

Changing the order of integration, if permissible, we get

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{\nu+\frac{1}{2}-s} \phi(s) ds \cdot 1/2\nu \int_0^{\infty} y^{\nu+\frac{1}{2}-s} \chi_{\nu, k, m}(\frac{1}{2} x^2 y^2) dy.$$

Evaluating the inner integral by (2.3A), we get

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s-\frac{1}{2}} \Gamma(x(v/2+s/2+1/4+m\pm m)) \Gamma(v/2-s/2+7/4+m\pm m)}{\Gamma(v/2+s/2+3/4+m-k) \Gamma(x(v/2+s/2+3/4+m\pm m))} \times \\ \times \phi(1-s)^{-s} x^s ds$$

which proves the result.

To justify the change in the order of integration in the integral, we observe that s and y -integrals converge absolutely under the conditions stated above. The repeated integral also

exists, since $\phi(s) = O\left(e^{-k|t|}\right)$, $k > 0$, $|t| \rightarrow \infty$.

Corollary: If $f(x)$ and $g(x)$ are $X_{v,k,m}$ -transforms of each other and

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \phi(s) x^{-s} ds,$$

$$\text{then } g(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s-\frac{1}{2}} \Gamma(x(v/2+s/2+1/4+m\pm m)) \Gamma(v/2-s/2+7/4+m-k)}{\Gamma(v/2+3/4+s/2+m-k) \Gamma(x(v/2+s/2+3/4+m\pm k))} \times \\ \times \phi(1-s)^{-s} x^s dx;$$

provided the conditions stated in the Theorem are satisfied.

$$\text{Ex. 1. Let } \phi(s) = 2^{\frac{1}{2}s + \frac{1}{4}\mu + \frac{1}{4}\nu - 5/4} \frac{\Gamma(\frac{1}{4} + \frac{1}{2}\mu + \frac{1}{2}s)}{\Gamma(\frac{1}{4} + \frac{1}{2}\nu + \frac{1}{2}s)},$$

given by Mohan³.

$$\text{Then } g(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s + \frac{1}{4}\mu + \frac{1}{4}\nu - 5/4} \frac{\Gamma(\frac{1}{4} + \frac{1}{2}\mu + \frac{1}{2}s)}{\Gamma(\frac{1}{4} + \frac{1}{2}\nu + \frac{1}{2}s)} x^{-s} ds, \quad (4.1A)$$

where $c > \mu + \nu - \lambda + 3/2$. Hence

$$g(x) = x^{\frac{1}{2}\mu + \frac{1}{2}\nu - \frac{1}{2}} e^{-\frac{1}{4}x^2} W_{1/2 + \mu/4 - \nu/4 - \frac{1}{2}\lambda, (\frac{1}{4}\mu - \nu)(\frac{1}{2}x^2)}.$$

If we write $\frac{1}{4}(\mu + \nu) = m$

and $\frac{1}{2} + \frac{1}{4}\mu + \frac{1}{4}\nu - \frac{1}{2}\lambda = k$, we get

$$g(x) = x^{\nu + 2m - \frac{1}{2}} e^{-\frac{1}{4}x^2} W_{k,m}(\frac{1}{2}x^2).$$

The corresponding value of $f(x)$ will be given by

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s-\frac{1}{2}} \Gamma(x(v/2+s/2+\frac{1}{4}+m\pm m)) \Gamma(v/2-s/2+7/4+m-k) \phi(1-s)x^{-s} ds}{\Gamma(v/2+3/4+s/2+m-k) \Gamma(x(v/2+3/4+s/2+m\pm m))}$$

Substituting the value of $\phi(1-s)$, we get

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s/2+\mu/4+v/4-5/4} \Gamma(x(v/2+s/2+1/4+m\pm m)) \Gamma(v/2-s/2+7/4+m-k) \times}{\Gamma(v/2+3/4+s/2+m-k) \Gamma(x(v/2+3/4+s/2+m\pm m))} \times \\ \frac{\Gamma(3/4+\frac{1}{2}\mu-\frac{1}{2}s) \Gamma(3/4+1/2v-\frac{1}{2}s)}{\Gamma(3/4+\frac{1}{2}\lambda-\frac{1}{2}s)} x^{-s} dx$$

Since $g(x)$ is Self-Reciprocal, hence $f(x) = g(x)$

I offer my grateful thanks to Prof. Brij Mohan for his help and guidance in the preparation of this note.

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THE ESSENTIAL OIL OF MENTHA RIPERITA VAR. VULGARIS

SOLE (Black mint)

By

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ABSTRACT

Mentha piperita var. *vulgaris* Sole (Black mint) has been successfully introduced in Jammu and Kashmir. The essential oil obtained from the locally raised plants on detailed chemical investigation was found to contain *l*-menthol 47.5%, menthone 11.6%, menthyl acetate 8.4%, *p*-cymene 1.9%, cadinene 1.6%, α -pinene 0.98% and limonene 0.96%.

A number of *Mentha* species grow in India, but none of them meet the official standards hence, the imported Peppermint oil is used in the Pharmaceutical and other industries. With a view to produce the Peppermint oil indigenously suckers of Black mint were procured from Royal Botanical Gardens, Kew, England for introduction in Jammu and Kashmir. Black mint has better vegetative growth and appears to be immune to rust attack.

Detailed chemical examination of the oil revealed the presence of *l*-menthol 47.5%, menthone 11.6%, menthyl acetate 8.4%, *p*-cymene 1.9%, cadinene 1.6%, β -pinene 0.98% and limonene 0.96%.

The physico-chemical constants of the oil under examination along with those of the European oil are given in table I given below :—

TABLE I

Particulars	Gildemeister and Hofmann I	Oil under examination
d_{15}^{15}	0.901 to 0.912	0.905
α_D	-20°0' to -33°0'	-30° 8'
n_D^{20}	1.460 to 1.463	1.461
A. V.	upto 1.6	0.06
Total Menthol content	43.5 to 68.0%	48.58%
Ester Menthol content	13.0 to 21.0%	9.0%
Menthone content	9.0 to 12.0%	12.98%
Solubility	Soluble in 2.0 to 3.5 vol. of 70% alcohol.	Soluble in 2 to 4 vols of 70% alcohol.

EXPERIMENTAL

The air dried leaves of *Mentha piperata* var. vulgaris (Black mint) raised here were steam distilled and gave 0.8 percent of the essential oil with characteristic odour. The oil was examined for its physicochemical properties. The results are given below :

$$d_{20}^{20} \ 0.901 ; n_D^{20} \ 1.461 ; \alpha_D \ -30.8' ; \text{Acid value } 0.06 ;$$

E. V. 25.6 ; E. V. after acetylation 181.4 ; Carbonyls (by oximation) 12.98.

CHROMATOGRAPHIC SEPARATION

40 g of the oil was chromatographed over 30 times Brockmann grade II alumina. Elution were successively made with petroleum ether, benzene, ether and ethyl acetate. The results of chromatography along with physical characteristics of the fractions obtained have been incorporated in table II given below :

TABLE II

No.	Eluate	Vol. of eluate (ml)	Wt. of fractions (g)	n_D^{20}
1	Petroleum ether	50	0.360	1.475
2	"	"	0.340	1.474
3	"	"	0.481	1.490
4	"	100	0.290	1.491
5	"	"	0.470	1.470
6	"	"	0.405	1.457
7	"	"	0.440	1.457
8	"	1000	0.340	1.446
9	Benzene	25	1.840	1.446
10	"	"	1.510	1.451
11	"	"	1.400	1.451
12	"	"	1.665	1.460
13	"	"	1.590	1.460
14	"	"	1.560	1.460
15	"	"	1.360	1.460
16	"	"	1.320	1.461
17	"	50	2.470	1.461
18	"	100	2.840	1.461
19	"	250	4.040	1.461
20	"	"	2.610	1.461
21	Ether	1000	0.150	1.470
22	Ethyl acetate	250	1.700	1.470

MIXING OF FRACTIONS

The fractions obtained by chromatography were suitably mixed on the basis of their physical characteristics. The mixed up fractions were further studied for their physical characteristics. The results are summarised below in table III.

TABLE III

Fractions	n_D^{20}	d_{20}^{20}	b. p. (C°)
A	1.475	0.8524	163—165
B	1.474	0.8523	177.5—178
C	1.491	0.8573	175—176
D	1.459	0.9245	260—273
E	1.446	0.9263	227—228
F	1.451	0.8946	209—210
G	1.460	0.8991	209—211
H	1.461	0.9003	...

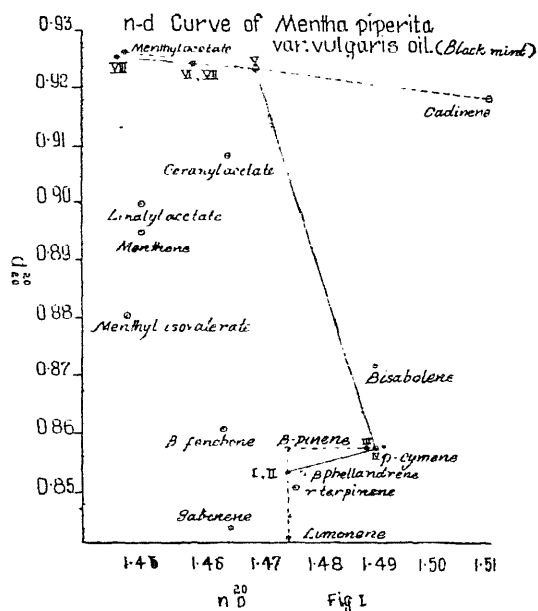


Fig. 1

IDENTIFICATION OF COMPONENTS

β -Pinene: Fraction 1 (A) (table II) appeared to be β -pinene on the basis of its physical properties. The n-d curve (Fig. 1) further supported this view. It

gave no precipitate with an alcoholic solution of mercuric acetate, even after 2-3 days, thus indicating the absence of α -pinene in it.

Limonene : Fraction 2 (B) (table II) was very close to limonene with respect to its physical properties as evident from the n-d-curve (fig. 1). It gave positive response to Libermann- Burchard test² for limonene. When treated with vapours of bromine in the ethereal solution, it gave tetrabromide m. p. 104-5°C. The tetrabromide showed no depression in the melting point on admixture with an authentic sample confirming the presence of limonene in the fraction.

p-cymene : Fractions 3 to 4 (C) (table II) were characterised as p-cymene through n-d-curve (Fig. I). It yielded p-hydroxyisopropyl- benzoic acid m. p. 153-155°C when oxidised with aqueous potassium permanganate as described by Wallach³.

Cadinene : Fractions 5 to 7 (D) (table II) were found to be mixture of cadinene and menthyl acetate as evident by the n-d-curve (Fig. I). It gave the colour reaction characteristic of cadinene as suggested by Wallach⁴.

Menthyl acetate : Fraction 8 to 9 (E) (table II) had a pleasant fruity odour, reminiscent of peppermint. It gave a positive spot test for acetate-esters as described by Feigl⁵. On saponification with 0.5 normal potassium hydroxide solution the fractions gave menthol which was identified by its further oxidation with Beckmann mixture. The oxidation product gave 2:4-dinitrophenyl hydrazone m.p. 145°C. Mixed melting point with an authentic sample showed no depression.

Menthone : Fractions 10 to 15 (F & G) (table II) resembled menthone in their physical properties. The identification was carried out by preparing their 2:4-dinitrophenyl hydrazone m.p. 145°C. In no case depression was observed when admixed with an authentic sample of 2:4-dinitrophenyl hydrazone which further confirmed its presence in the fractions.

l-Menthol : Fractions 16 to 22 (H) (table II) had a powerful peppermint like odour and a cooling taste. The physical properties confirmed with the values reported in literature. On oxidation with Beckmann mixture it yielded menthone which gave 2:4-dinitrophenyl hydrazone m.p. 145°C. No change in the melting point was observed when a mixed melting point was taken with an authentic sample of l-menthone 2:4-dinitrophenyl hydrazone.

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SOME KERNEL FUNCTIONS

By

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ABSTRACT

A generalisation of the Hankel transform has been given by Mehra (2) in the form

$$g(x) = \left(\frac{1}{2}\right)^v \int_0^\infty (xy)^{v+\frac{1}{2}} \chi_{v, k+\frac{1}{2}, m} (1/4 x^2 y^2) f(y) dy,$$

In this paper we have obtained a number of kernels with the help of two main theorems given by Mehra (2).

1. A generalisation of the Hankel transform

$$g(x) = \int_0^\infty \sqrt{xy} J_v(xy) f(y) dy, \quad (1.1)$$

has been given by A. N. Mehra [2] in the form

$$g(x) = \frac{1}{2^v} \int_0^\infty (xy)^{v+\frac{1}{2}} \chi_{v, k+\frac{1}{2}, m} (1/4 x^2 y^2) f(y) dy, \quad (1.2)$$

where

$$\begin{aligned} \chi_{v, k+\frac{1}{2}, m}(x) = & \sum_{m, -m} \frac{\Gamma(2m)}{\Gamma(m-k)} \frac{\Gamma(v+1-m-3k)}{\Gamma(v+1-M-2k\pm m)} x^{k-m} \times \\ & \times {}_2F_3 \left(\begin{matrix} 1+k-m, v+1-m-3k \\ 1-2m, v+1-m-2k\pm m \end{matrix} \right). \end{aligned} \quad (1.3)$$

He calls $g(x)$ in (1.2) to be the $\chi_{v, k+\frac{1}{2}, m}$ -transform of $f(x)$ and has shown that the reciprocity formula holds for this transform also, i.e. if $g(x)$ is the $\chi_{v, k+\frac{1}{2}, m}$ -transform of $f(x)$, then so is $f(x)$ of $g(x)$. In case $g(x)=f(x)$ i.e. if $f(x)$ is its own $\chi_{v, k+\frac{1}{2}, m}$ -transform, we say that $f(x)$ is self-reciprocal in the $\chi_{v, k+\frac{1}{2}, m}$ -transform or that $f(x)$ is $R_v(k+\frac{1}{2}, m)$. It may be noted that on putting $k=-m$, (1.2) reduces to (1.1).

The object of this paper is to find out certain kernels by using two theorems of Mehra [2] given in the following section. The importance of these kernels lies in the fact that we get a number of kernels deduced from these as particular cases by suitable substitutions.

2. We shall use the following two theorems of Mehra [2] :

Theorem 1. If $f(x)$ is $R_\nu(k + \frac{1}{2}, m)$ and

$$P(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^s \Gamma_x(\nu/2 + s/2 + 1/4 - k \pm m) \Gamma_x(\mu/2 + s/2 + 1/4 - l \pm n)}{\Gamma(\nu/2 + s/2 + 1/4 - 2k) \Gamma(\mu/2 + s/2 + 1/4 - 2l)} \bar{\omega}(s) x^{-s} ds, \quad (2.1)$$

where $0 < c < 1$ and $\bar{\omega}(s) = \bar{\omega}(1-s)$,

(2.2)

$$\text{then } g(x) = \int_0^\infty f(y) P(xy) dy$$

(2.3)

is $R_\mu(l + \frac{1}{2}, n)$. In other words, $P(x)$ is a kernel transforming $R_\nu(k + \frac{1}{2}, m)$ into $R_\mu(l + \frac{1}{2}, n)$.

Since the result is symmetrical in μ, ν, k, l and m, n ; a kernel $P(x)$ which transforms $R_\nu(k + \frac{1}{2}, m)$ into $R_\mu(l + \frac{1}{2}, n)$ also effects the converse transformation.

Theorem 2. If $f(x)$ is $R_\nu(k + \frac{1}{2}, m)$ and

$$P(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma_x(\nu/2 + s/2 + 1/4 - k \pm m) \Gamma_x(\mu/2 - s/2 + \frac{3}{4} - l \pm n)}{\Gamma(\nu/2 + s/2 + 1/4 - 2k) \Gamma(\mu/2 - s/2 + \frac{3}{4} - 2l)} \bar{\omega}(s) x^{-s} ds,$$

where $0 < c < 1$ and $\bar{\omega}(s) = \bar{\omega}(1-s)$ (2.4)

$$\text{then } g(x) = \int_0^\infty f(xy) P(y) dy \quad (2.5)$$

is $R_\mu(l + \frac{1}{2}, n)$. In other words, $P(x)$ is a kernel transforming $R_\nu(k + \frac{1}{2}, m)$ into $R_\mu(l + \frac{1}{2}, n)$.

3. Consider the function

$$\sqrt{x}^{-1} G_{p+2, p+4}^{m+4, m} \left(\frac{x^2}{4} \left| \begin{matrix} a_1, \dots, a_p, \nu/2 - 2k, \mu/2 - 2l \\ \nu/2 - k - m, \nu/2 - k + m, \mu/2 - l \pm n, b_1, \dots, b_p \end{matrix} \right. \right).$$

where $m < p < 2m+1$.

By definition (II), p. 207 this equals

$$\frac{\sqrt{x}}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{\prod_{j=1}^m \Gamma(b_j - t) \prod_{j=1}^m \Gamma(1 - a_j + t) \Gamma_x(\mu/2 - l \pm n - t)}{\prod_{j=m+1}^p \Gamma(1 - b_j + t) \prod_{j=m+1}^p \Gamma(a_j - t) \Gamma(\nu/2 - 2k - t)} \frac{\Gamma_x(\nu/2 - t - k \pm m)}{\Gamma(\mu/2 - 2l - t)} \left(\frac{x^2}{4}\right)^t dt.$$

Note $\Gamma_x(a \pm b) = \Gamma(a+b) \Gamma(a-b)$; $a \pm b = (a+b)(a-b)$.

On replacing t by $-s - \frac{1}{2}$, we get the function

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^s \prod_{j=1}^m \Gamma(\frac{1}{2} + b_j + s/2) \prod_{j=1}^m \Gamma(3/4 - a_j - s/2) \Gamma_x(\mu/2 + s/2 + \frac{1}{2} - l \pm n)}{\sqrt{2} \prod_{j=m+1}^p \Gamma(3/4 - b_j - s/2) \prod_{j=m+1}^p \Gamma(\frac{1}{2} + a_j + s/2) \Gamma(\nu/2 + s/2 + \frac{1}{2} - 2k)} \frac{\Gamma_x(\nu/2 + \frac{1}{2} + s/2 - k \pm m)}{\Gamma(\mu/2 + s/2 + \frac{1}{2} - 2l)} x^{-s} ds.$$

This is of the same form as (2.1) with $0 < c < 1$ and

$$\omega(s) = \frac{\prod_{j=1}^m \Gamma(\frac{1}{2} + b_j + s/2) \prod_{j=1}^m \Gamma(3/4 - a_j - s/2)}{\sqrt{2} \prod_{j=m+1}^p \Gamma(3/4 - b_j - s/2) \prod_{j=m+1}^p \Gamma(1/4 + a_j + s/2)}$$

This satisfies the functional relation (2.2) if

$$a_j + b_j = 0, j=1, 2, \dots, p.$$

Hence we find that

$$\sqrt{x} G_{\substack{m+4, m \\ p+2, p+4}} \left(\frac{x^2}{4} \middle| \begin{matrix} -b_1, \dots, -b_p \\ \nu/2 - k \pm m, \mu/2 - l \pm n, b_1, \dots, b_p \end{matrix} \right), \quad m < p < 2m+1, \quad (3.1)$$

is a kernel which transforms $R_\nu(k + \frac{1}{2}, m)$ into $R_\mu(l + \frac{1}{2}, n)$ and vice versa in accordance with equation (2.3).

Particular cases: Giving suitable values to the parameters in (3.1) we can deduce a number of kernels. In the following example A is a constant multiplier independent of x everywhere.

$$(i) x^{v-2k} e^{-\frac{1}{2}x} W_{2k+\frac{1}{2}, 2m}(x)$$

$$\equiv A \sqrt{x} G_{2,4}^{4,0} \left(\frac{x^2}{2} \middle| \begin{matrix} v/2-2k, \mu/2-2l \\ v/2-k \pm m, \mu/2-l \pm n \end{matrix} \right).$$

is a kernel transforming $R_v(k+\frac{1}{2}, m)$ into $R_{v+1}(k+\frac{1}{2}, m)$.

$$(ii) x^{v+\frac{1}{2}} \chi_{\frac{\mu+v}{2}, k+\frac{1}{2}, m} \left(\frac{x^2}{4} \right)$$

$$\equiv A \sqrt{x} G_{57}^{51} \left(\frac{x^2}{4} \middle| \begin{matrix} -\mu/2+2k, \mu/2-k \pm m, v/2-2l, \mu/2-2k \\ \mu/2-k \pm m, v/2-l \pm n, \mu/2-2k, -\mu/2-k \pm m \end{matrix} \right).$$

is a kernel transforming a $R_\mu(k+\frac{1}{2}, m)$ into $R_v(k+\frac{1}{2}, m)$.

$$(iii) x^{v+2m+\frac{1}{2}} k_{m+n}(x/2) k_{m+n}(x/2)$$

$$\equiv A \sqrt{x} G_{24}^{40} \left(\frac{x^2}{4} \middle| \begin{matrix} v/2+m+\frac{1}{2}, v/2+m \\ v/2, v/2+m-n, v/2+2m, v/2+m+n \end{matrix} \right)$$

is a kernel transforming $R_v(\frac{1}{2}, m)$ into $R_\mu(0, n)$,

where $\mu = v+2m-2n$, [2, p. 288].

$$(iv) x^{\mu-\frac{1}{2}} W_{k-m-\frac{1}{2}, -2m-\frac{1}{2}}(x) W_{-k+m+\frac{1}{2}, -2m-\frac{1}{2}}(x)$$

$$\equiv A \sqrt{x} G_{24}^{40} \left(\frac{x^2}{4} \middle| \begin{matrix} \mu/2-m+k, \mu/2+1+m-k \\ \mu/2, \mu+1/2, \mu/2-\frac{1}{2}-2m, \mu/2+\frac{1}{2}+2m \end{matrix} \right)$$

is a kernel transforming $R_\mu(k, m)$ into $R_{\mu+1}(\frac{1}{2}-k, -m-\frac{1}{2})$.

$$(v) x^{v+\frac{1}{2}} \chi_{\frac{\mu+v}{2}, k, m} \left(\frac{x^2}{4} \right)$$

$$\equiv A \sqrt{x} G_{57}^{51} \left(\frac{x^2}{4} \middle| \begin{matrix} k-m-\frac{1}{2}-\mu/2, \mu/2, \mu/2+2m, v/2-k+m+\frac{1}{2}, \mu/2+\frac{1}{2}+m-k \\ \mu/2, v/2, \mu/2+2m, v/2+2n, \mu/2+\frac{1}{2}+m-k, -\mu/2, -\mu/2-2m \end{matrix} \right).$$

is a kernel transforming $R_\mu(k, m)$ into $R_v(k, m)$, [2, p. 289].

$$(vi) x^{v+2m} e^{-\frac{1}{2}x} W_{2k-\frac{1}{2}, -2m}(x)$$

$$\equiv A \sqrt{x} G_{24}^{40} \left(\frac{x^2}{4} \left| \begin{matrix} \nu/2 + \frac{1}{2} + m - k, \nu/2 + 1 + k - m \\ \nu/2, \frac{\nu+1}{2}, \frac{1}{2} + \nu/2 + 2m, \nu/2 + 2m \end{matrix} \right. \right)$$

is a kernel transforming $R_\nu(k, m)$ into $R_{\nu+1}(k, m)$. [2, p. 288].

4. Consider the function

$$\sqrt{x} G_{pp}^{mm} \left(x^2 \left| \begin{matrix} -\mu/2 + l + n, \mu/2 - l - n, a_3, \dots, a_{p-1}, \nu/2 - 2k \\ \nu/2 - k \pm m, b_3, \dots, b_{p-1}, \mu/2 + 2l \end{matrix} \right. \right)$$

By definition [(1), p. 207] this function

$$\begin{aligned} &= \frac{\sqrt{x}}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{\prod_{J=3}^m \Gamma(\nu/2 - k \pm m - t) \prod_{J=3}^m \Gamma(b_j - t) \Gamma(1 + \mu/2 - l \pm n - t)}{\prod_{J=m+1}^{p-1} \Gamma(1 - b_j + t) \Gamma(1 + \mu/2 - 2b + t) \prod_{J=m+1}^{p-1} \Gamma(a_j - t)} \times \\ &\quad \times \frac{\prod_{J=3}^m \Gamma(1 - a_j + t)}{\Gamma(\nu/2 - 2k - t)} \cdot (x^2)^t dt. \end{aligned}$$

On replacing t by $-s - \frac{1}{4}$, we get the function

$$\begin{aligned} &\frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \frac{\prod_{J=3}^m \Gamma(\nu/2 + s/2 + 1/4 - k \pm m) \prod_{J=3}^m \Gamma(1/4 + b_j + s/2) \prod_{J=3}^m \Gamma(3/4 + \mu/2 - s/2 - l \pm n)}{\prod_{J=m+1}^{p-1} \Gamma(3/4 - b_j - s/2) \Gamma(\mu/2 - 2l - s/2 + 3/4) \prod_{J=m+1}^{p-1} \Gamma(a_j + s/2 + 1/4)} \times \\ &\quad \times \frac{\prod_{J=3}^m \Gamma(3/4 - a_j - s/2)}{\Gamma(\nu/2 - 2k + s/2 + 1/4)} x^{-s} ds. \end{aligned}$$

This is of the same form as (2.4) with $0 < c < 1$,

where

$$\omega(s) = \frac{\prod_{J=3}^m \Gamma(\frac{1}{4} + b_j + s/2) \prod_{J=3}^m \Gamma(\frac{3}{4} - a_j - s/2)}{\prod_{J=m+1}^{p-1} \Gamma(\frac{3}{4} - b_j - s/2) \prod_{J=m+1}^{p-1} \Gamma(\frac{1}{4} + a_j + s/2)},$$

which satisfies the functional relation (2.5) if

$$a_j + b_j = 0, \quad J = 3, 4, \dots, p-1.$$

Hence we find that

$$\sqrt{x}^{-} G_{p\ p}^{mm} \left(x^2 \left| \begin{array}{l} -\mu/2+l\pm n, -b_3, \dots, -b_{p-1}, v/2-2k \\ v/2-k\pm m, b_3, \dots, b_{p-1}, -\mu/2+2l \end{array} \right. \right) \quad (4.1)$$

$$2m > p > m,$$

is a kernel transforming $R_\nu (k+\frac{1}{2}, m)$ into $R_\mu (l+\frac{1}{2}, n)$.

4.1. *Particular cases*: Giving suitable values to the parameters in (4.1) we can again deduce some kernels as particular cases. A is a constant multiplier independent of x everywhere in the following examples.

$$\sqrt{x}^{-} G_{33}^{22} \left(x^2 \left| \begin{array}{l} -\frac{1}{2}\mu+l\pm n, \frac{1}{2}v-2k \\ v/2-k\pm m, -\mu/2+2l \end{array} \right. \right),$$

is a kernel transforming $R_\nu (k+\frac{1}{2}, m)$ into $R_\mu (l+\frac{1}{2}, n)$.

$$(ii) {}_2F_1 \left[\begin{array}{l} v/2+2m+2n+\frac{3}{4}, \frac{1}{2} \\ v/2+2m+5/4+n-l \end{array} ; -x^2 \right]$$

$$\equiv A \sqrt{x}^{-} G_{55}^{33} \left(x^2 \left| \begin{array}{l} -\frac{v+4m}{2}, -\frac{v+4m}{2}-2n, \frac{1}{4}, \frac{v+4m}{2}, v/2 \\ v/2, \frac{v+4m}{2}, -\frac{1}{4}, -\frac{v+4m}{2}, -\frac{v+4m}{2}-\frac{1}{2}-n+l \end{array} \right. \right)$$

is a kernel transforming $R_\nu (\frac{1}{2}+m, m)$ into $R_{\nu+4m} (l, n)$.

$$(iii) \sqrt{x}^{-} G_{33}^{22} \left(x^2 \left| \begin{array}{l} -\mu/2, -\mu/2-2n, v/2+\frac{1}{2}+m-k \\ v/2, v/2+2m, -\mu/2-\frac{1}{2}-n-l \end{array} \right. \right)$$

is a kernel transforming $R_\nu (k, m)$ in $R_\mu (l, n)$, [2, p. 290].

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ON THE GENERALISED HANKEL TRANSFORMS

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ABSTRACT

Agarwal (1) gave a generalisation of Hankel transform in the form

$$f(x) = \left(\frac{1}{2}\right)^\gamma \int_0^\infty (xy)^{\gamma+\frac{1}{2}} J_\gamma^\mu \left(\frac{1}{2} x^2 y^2\right) g(y) dy.$$

Some properties to this transform have been given by Agarwal. In this present paper some properties in the form of theorems are given which are mainly helpful for evaluating infinite integrals. The enunciation of one of the theorems is as follows :—

Theorem 1 : If $\phi(s)$ is the Meijer-transform of $f(x)$ and $f(x)$ is the J_γ^μ -transform of $g(x)$, then

$$\phi(s) = \int_0^\infty g(y) \psi(y) dy,$$

$$\text{when } \psi(y) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \frac{(y/s)^{\frac{1}{2}+\gamma+2r}}{2^{\gamma+2r}} \frac{\Gamma_x(\gamma+2r-k+\frac{1}{2}\pm m)}{\Gamma(1+\gamma+\mu r) \Gamma(\gamma+2r-2k+3/2)},$$

provided that the integrals involved converge absolutely,

$$R(\gamma+\frac{1}{2}-k\pm m) > 0 \text{ and } 0 < \mu \leq 1.$$

1. Agarwal [1] gave a generalisation of the Hankel transform by means of the integral equation.

$$f(x) = \left(\frac{1}{2}\right)^\gamma \int_0^\infty (xy)^{\gamma+\frac{1}{2}} J_\gamma^\mu \left(\frac{1}{2} x^2 y^2\right) g(y) dy \quad (1.1)$$

$$\text{where } J_\gamma^\mu(x) = \sum_{r=0}^{\infty} \frac{(-x)^r}{r! \Gamma(1+\gamma+\mu r)}, \mu > 0 \quad (1.2)$$

Some well-known properties and certain inversion formulae to this transform has been investigated by him [2].

Ram Kumar [4,5] has made a notable contribution to this field by giving a transform in a more convenient form.

The object of the paper is to investigate some properties of this transform and apply them in evaluating certain infinite integrals.

2. *Theorem 1.* If $\phi(s)$ is the Meijer-transform of $f(x)$ and $f(x)$ is the J_γ^μ -transform of $g(x)$, then

$$\phi(s) = \int_0^\infty g(y) \psi(y) dy, \quad (2.1)$$

where

$$\psi(y) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \frac{\Gamma_x(\gamma+2r+3/2-k \pm m) (y/s)^{\gamma+\frac{1}{4}+2r}}{\Gamma(1+\gamma+\mu r) \Gamma(\gamma+2r+3/2-2k) 2^{\gamma+2r}} * \quad (2.2)$$

provided that the integrals involved are absolutely convergent and $\text{Re}(\gamma+3/2-k \pm m) > 0$ and $0 < \mu \leq 1$.

Proof: The Meijer transform of $\phi(s)$ will be given by

$$\phi(s) = s \int_0^\infty (sx)^{-k-\frac{1}{2}} e^{-\frac{1}{2}sx} W_{k+\frac{1}{2},m}(sx) f(x) dx, \quad (2.3)$$

$$\text{and } f(x) = \left(\frac{1}{2}\right)^\gamma \int_0^\infty (xy)^{\gamma+\frac{1}{2}} J_\gamma^\mu\left(\frac{1}{4}x^2 y^2\right) g(y) dy. \quad (2.4)$$

Substituting the value of $f(x)$ from (2.4) in (2.3) we get

$$\phi(s) = s \int_0^\infty (sx)^{-k-\frac{1}{2}} e^{-\frac{1}{2}sx} W_{k+\frac{1}{2},m}(sx) \left(\frac{1}{2}\right)^\gamma \int_0^\infty (xy)^{\gamma+\frac{1}{2}} J_\gamma^\mu\left(\frac{1}{4}x^2 y^2\right) g(y) dy.$$

Changing the order of integration, which is justifiable, we get

$$\phi(s) = \left(\frac{1}{2}\right)^\gamma s \int_0^\infty y^{\gamma+\frac{1}{2}} g(y) dy \int_0^\infty e^{-\frac{1}{2}sx} (sx)^{-k-\frac{1}{2}} W_{k+\frac{1}{2},m}(sx) x^{\gamma+\frac{1}{2}} J_\gamma^\mu\left(\frac{1}{4}x^2 y^2\right) dx.$$

Expanding $J_\gamma^\mu(x)$ in ascending powers of x and then evaluating the integral with the help of an integral due to Goldstein [3], viz,

$$\int_0^\infty x^{l-1} e^{-\frac{1}{2}sx} W_{k,m}(x) dx = \frac{\Gamma_x(l+\frac{1}{2} \pm m)}{\Gamma(l-k+1)}, \quad \text{Re}(l+\frac{1}{2} \pm m) > 0,$$

we get

$$\phi(s) = \int_0^\infty g(y) \psi(y) dy,$$

* $\Gamma(a \pm b) = \Gamma(a+b) \Gamma(a-b)$.

where

$$\psi(y) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \frac{\Gamma_x(\gamma+2r+3/2-k\pm m)}{\Gamma(1+\gamma+\mu r) \Gamma(\gamma+2r+3/2-2k)} \frac{(y/s)^{\gamma+\frac{1}{2}+2r}}{2^{\gamma+2r}},$$

provided that $R(\gamma+3/2-k\pm m) > 0$ and $0 < \mu \leq 1$, and the integrals involved converge absolutely.

The change in the order of integration will be justified if the integrals

$$(i) \int_0^{\infty} (xy)^{\gamma+\frac{1}{2}} J_{\gamma}^{\mu}(\tfrac{1}{4}x^2y^2) g(y) dy,$$

and

$$(ii) \int_0^{\infty} e^{-\frac{1}{2}sx} (sx)^{-k-\frac{1}{2}} W_{k+\frac{1}{2},m}(sx) x^{\gamma+\frac{1}{2}} J_{\gamma}^{\mu}(\tfrac{1}{4}x^2y^2) dx,$$

are absolutely convergent. Assuming (i) to be absolutely convergent, (ii) converges absolutely if $R(\gamma+3/2-k\pm m) > 0$, and $0 < \mu \leq 1$,

In order to illustrate the theorem the following example is worth mention.

Example. Let $f(x) = (\tfrac{1}{2})^{\gamma+1} x^{\gamma+\frac{1}{2}} \sum_{r=0}^{\infty} \frac{\Gamma(r+\gamma/2+3/4)}{r! \Gamma(1+\gamma+\mu r)} (-x^2/4)^r,$

and $g(y) = e^{-y^2},$

where $0 < \mu \leq 1$ and $R(\gamma) > -3/2$.

Now,

$$\begin{aligned} \phi(s) &= \int_0^{\infty} g(y) \psi(y) dy \\ &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \frac{\Gamma_x(\gamma-k+2r\pm m+3/2)}{\Gamma(1+\gamma+\mu r) \Gamma(\gamma+2r-2k+3/2)} \frac{s^{-(\frac{1}{2}+\gamma+2r)}}{2^{\gamma+2r}} \int_0^{\infty} y^{\frac{1}{2}+\gamma+2r} e^{-y^2} dy. \end{aligned}$$

Integrating term by term, which is permissible, we get

$$\phi(s) = \frac{1}{2^{\gamma+1}} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \frac{\Gamma_x(\gamma+2r+3/2-k\pm m)}{\Gamma(1+\gamma+\mu r) \Gamma(\gamma+2r-2k+3/2)} \frac{s^{-(\frac{1}{2}+\gamma+2r)}}{2^{2r}}.$$

provided that $0 < \mu \leq 1$, $R(\gamma) > -3/2$, $R(\gamma + 3/2 - k \pm m) > 0$.

Also

$$\begin{aligned}\phi(s) &= s \int_0^\infty (sx)^{-k-\frac{1}{2}} e^{-\frac{1}{2}sx} W_{k+\frac{1}{2},m}(sx) \left(\frac{1}{2}\right)^{\gamma+1} x^{\gamma+\frac{1}{2}} \times \\ &\quad \times \sum_{r=0}^\infty \frac{\Gamma(\gamma/2 + \frac{3}{4} + r)}{r! \Gamma(1 + \gamma + \mu r)} \left(-\frac{x^2}{4}\right)^r dx, \\ &= \sum_{r=0}^\infty \frac{(-1)^r}{r!} \frac{\Gamma(\gamma + \gamma/2 + \frac{3}{4})}{\Gamma(1 + \gamma + \mu r)} \left(\frac{1}{2}\right)^{1+\gamma+2r} s \int_0^\infty (sx)^{-k-\frac{1}{2}} e^{-\frac{1}{2}sx} W_{k+\frac{1}{2},m}(sx) \\ &\quad \times x^{\frac{1}{2}+\gamma+2r} dx, \\ &= \sum_{r=0}^\infty \frac{(-1)^r}{r!} \frac{\Gamma(\gamma/2 + \frac{3}{4} + r)}{\Gamma(1 + \gamma + \mu r)} \left(\frac{1}{2}\right)^{1+\gamma+2r} s^{-(\frac{1}{2}+\gamma+2r)} \int_0^\infty u^{\gamma-k+2r} e^{-\frac{1}{2}u} \times \\ &\quad \times W_{k+\frac{1}{2},m}(u) du, \\ &= \sum_{r=0}^\infty \frac{(-1)^r}{r!} \frac{\Gamma(r + \gamma/2 + \frac{3}{4})}{\Gamma(1 + \gamma + \mu r)} \frac{s^{-\frac{1}{2}+\gamma+2r} \Gamma_x(\gamma-k+2r+3/2 \pm m)}{2^{1+\gamma+2r} \Gamma(\gamma+2r-2k+3/2)},\end{aligned}$$

provided that $R(\gamma - k + 3/2 \pm m) > 0$, $R(\gamma) > -3/2$ and $0 < \mu \leq 1$.

Thus the theorem is verified.

3. **Theorem 2** :—If $\phi(s)$ be the generalised Laplace transform of $f(x)$ namely

$$\phi(s) = s \int_0^\infty e^{-\frac{1}{2}sx} (sx)^{m-\frac{1}{2}} W_{k,m}(sx) f(x) dx,$$

given by Varma [9]

and $f(x)$ is the J_γ^μ - transform of $g(x)$ then

$$\phi(s) = \int_0^\infty g(y) \psi(y) dy$$

where

$$\psi(y) = \left(\frac{1}{2}\right)^\gamma s^{-\gamma-\frac{1}{2}} \sum_{r=0}^\infty \frac{(-1)^r \left(\frac{1}{2}s\right)^{1/2r} \Gamma_x(\gamma+m+2r+3/2 \pm m)}{r! \Gamma(1+r+\mu r) \Gamma(\gamma+m+2r-k+2)} y^{\gamma+\frac{1}{2}+2r}$$

provided that $0 < \mu \leq 1$ and $R(\gamma + m + 3/2 \pm m) > 0$ and the integrals involved converge absolutely.

Proof of this theorem is similar to that of Theorem 1.

In order to illustrate the theorem the following example is worth mentioning :

$$\text{Let } g(x) = x^{\frac{1}{2}} k_{\gamma}(x),$$

$$\text{and } f(x) = x^{\gamma + \frac{1}{2}} \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(1+\gamma+r) (-x^2)^r}{\Gamma(1+\gamma+\mu r)}, \quad 0 < \mu \leq 1 \quad R(\gamma) > -1.$$

$$\text{Therefore } \phi(s) = \int_0^{\infty} g(y) \psi(y) dy$$

$$= \left(\frac{1}{2}\right)^{\gamma} s^{-\gamma - \frac{1}{2}} \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{1}{2}s\right)^{2r} \Gamma_x(\gamma + 2r + m + 3/2 \pm m)}{\Gamma(1+\gamma+\mu r) \Gamma(\gamma + 2r + 2 - k + m)} \int_0^{\infty} y^{\gamma + 2r + 1} k_{\gamma}(y) dy.$$

Evaluating the integral, we get

$$\phi(s) = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(1+\gamma+r) \Gamma_x(\gamma + 2r + 3/2 + m \pm m)}{(s)^{\frac{1}{2} + \gamma + 2r} \Gamma(1+\gamma+\mu r) \Gamma(\gamma + 2r + 2 - k + m)}.$$

Also,

$$\phi(s) = s \int_0^{\infty} e^{-\frac{1}{2}sx} (sx)^{m - \frac{1}{2}} W_{k,m}(sx) \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(1+\gamma+r)}{\Gamma(1+\gamma+\mu r)} x^{\frac{1}{2} + \gamma + 2r} dx.$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r s^{-(\frac{1}{2} + \gamma + 2r)} \Gamma(1+\gamma+r)}{\Gamma(1+\gamma+\mu r)} \int_0^{\infty} u^{\gamma + m + 2r} e^{-\frac{1}{2}u} W_{k,m}(u) dx.$$

Evaluating the integral we get

$$\phi(s) = \sum_{r=0}^{\infty} \frac{(-1)^r s^{-(\frac{1}{2} + \gamma + 2r)} \Gamma(1+\gamma+r) \Gamma_x(\gamma + m + 2r + 3/2 \pm m)}{\Gamma(1+\gamma+\mu r) \Gamma(\gamma + m + 2r + 2 - k)}.$$

Thus the theorem is verified.

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EFFECT OF SOIL AND FOLIAR APPLICATION OF NITROGEN ON THE CHEMICAL COMPOSITION OF PENNISETUM TYPHOTDEUM (BAJRA)

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ABSTRACT

1. Protein values of bajra seeds improved due to the nitrogenous fertilization either through soil or leaves.
2. Fat content of bajra was not affected due to manuring.

Bajra is predominantly a crop suited to light soils and regions of low rainfall and can be grown in tracts receiving a rainfall of 16 to 20 inches per annum. Bajra is an important food crop in Maharashtra, Gujarat, Rajasthan, Madhya Pradesh, Andhra Pradesh and some parts of Western Uttar Pradesh. The quality of Bajra like any other crop is governed by climatic conditions, soil differentiations and agronomical practices.

Agricultural Chemists Madras State (Anon 1956) observed maximum amount of nitrogen and potash in the bajra grains from Dhaincha green manured plots while those from compost treatment contained the highest amount of phosphoric acid thus establishing the superiority of green manure. Chavan (1958) concluded higher albuminoids, fat and mineral content in Bajra grain than the jowar grains. Mehta and Shah (1958) found less variation in potassium, calcium and magnesium contents in bajra grains than in stalks with leaves. Phosphatic manures increased the phytin phosphorus and total phosphorus while cattle urine decreased them in bajra grains hence the grains from phosphate fertilized plots were inferior in nutritative value than from cattle manure plots (Anon 1956). Absence of linolenic acid in the bajra fat was observed at Regional Research Centre, PIRCCOM, Kanpur (1962). Bajra is reported to have a higher fat content than other millets (Krishna Swamy, 1962). The present investigations were made to see the effect of nitrogen applied through leaves or soil on the quality of Bajra grain.

MATERIAL AND METHODS

Bajra seed samples were collected from each plot at the time of harvesting the crop. These were cleaned of any other foreign material and stored in air tight bottles.

Moisture and protein were determined by A. O. A. C. methods (1955); fat by extracting the ground seeds with light petroleum ether in Soxhlet's apparatus. Ash and crude fibre were determined by A. O. A. C. (1955) method, carbohydrates were determined difference.

DATA AND DISCUSSION

Moisture, fat and protein contents were determined in the seed samples of each plot and are given in table I. Ash, crude fibre and carbohydrates were determined in composite samples of each treatment and are reported in table II.

TABLE I
Moisture, Fat and Protein content in Bajra as affected by soil and foliar
application of nitrogen

Treatments		Replications				Average
		1	2	3	4	
1. 0 lb. N/acre Foliar	Moisture%	13.06	12.82	12.24	12.03	12.54
	Fat%	4.15	4.26	4.98	5.05	4.66
	Protein	7.63	7.40	8.30	8.31	7.91
2. 10 lb. N/acre Foliar	Moisture%	12.94	12.70	12.61	10.93	12.29
	Fat%	4.52	4.78	4.04	4.73	4.67
	Protein%	7.81	8.29	8.28	8.35	8.18
3. 20 lb. N/acre Foliar	Moisture%	13.27	12.51	12.28	11.55	12.40
	Fat%	4.82	4.51	4.08	4.97	4.60
	Protein%	7.38	8.31	7.85	9.03	8.14
4. 0 lb. N/acre Soil	Moisture%	12.84	12.83	12.40	10.98	12.26
	Fat%	4.57	4.15	5.07	4.95	4.60
	Protein%	8.06	7.41	8.09	8.13	7.92
5. 20 lb. N/acre Soil	Moisture%	12.68	13.21	12.38	10.62	12.22
	Fat%	4.60	4.29	4.82	4.76	4.62
	Protein%	8.13	7.87	8.31	7.88	8.27
6. 40 lb. N/acre Soil	Moisture%	12.70	12.20	12.41	11.21	12.13
	Fat%	4.46	4.67	4.44	4.92	4.62
	Protein%	8.29	7.63	7.84	6.33	8.02

Correlation Factor :—

Moisture and Fat = 0.7708

Moisture and Protein = 0.3248

Fat and Protein = 0.4041

TABLE II

Ash, Crude fibre and Soluble carbohydrates in Bajra as affected by soil and foliar application of nitrogen

Treatments	Ash%	Crude Fibre%	Soluble carbohydrates%
1. 0 lb. N/acre Foliar	1.58	2.47	70.01
2. 10 lb. N/acre Foliar	1.58	2.67	70.91
3. 20 lb. N/acre Foliar	1.62	2.27	71.02
4. 0 lb. N/acre Soil	1.63	2.23	70.02
5. 20 lb. N/acre Soil	1.60	2.40	71.24
6. 40 lb. N/acre Soil	1.60	2.39	71.24
Protein and Carbohydrates	=	0.8026	
Protein and Fibre	=	0.2980	
Fibre and Carbohydrates	=	0.1103	
Ash and Fibre	=	0.5518	

The variation in range of moisture and fat (Table I) had been narrow and do not appear to have any relationship with either forms of manuring. Higher protein content was recorded in seeds where nitrogen was applied either through leaves or through soil. As regards protein content higher dose of 20 lb. nitrogen through leaves or 40 lb. nitrogen through soil did not prove superior over 10 lb. nitrogen through leaves or 20 lb nitrogen through soil. However soil application of nitrogen showed highest protein content. The application of nitrogenous manures did not suppress the fat content. No negative correlation was found between fat and protein contents as in the case of oilseeds which is an interesting observation.

The ash and crude fibre (Table II) contents were not affected due to nitrogenous fertilization. Their variation range is also very narrow. Higher soluble carbohydrates were noted in seed samples where nitrogen was applied. The interesting observation that the fat content of Bajra is not affected due to nitrogenous manuring while the protein value improves was confirmed in these studies also.

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CERTAIN KERNELS IN THE GENERALISED HANKEL TRANSFORMS

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ABSTRACT

A generalisation of the Hankel transform

$$g(x) = \int_0^{\infty} \sqrt{xy} J_{\nu}(xy) f(y) dy,$$

has been recently introduced by Roop Narain (4) in the form

$$g(x) = \left(\frac{1}{2}\right)^{\nu} \int_0^{\infty} (xy)^{\nu+\frac{1}{2}} \chi_{\nu, k, m} \left(\frac{1}{2} x^2 y^2\right) f(y) dy.$$

He has established certain kernels in this transform, connecting different classes of self-reciprocal functions. Some more new kernels have been investigated by R. K. Saxena (5). In this paper some kernels, following in the main the methods given by R. P. Agarwal, have been investigated. These kernels are the resultant of two kernels and the resultant of two self-reciprocal functions.

1. A generalisation of the Hankel transform

$$g(x) = \int_0^{\infty} \sqrt{xy} J_{\nu}(xy) f(y) dy, \tag{1.1}$$

has been recently introduced by Roop Narain in 1956-57 in the form

$$g(x) = 1/2^{\nu} \int_0^{\infty} (xy)^{\nu+\frac{1}{2}} \chi_{\nu, k, m} \left(\frac{1}{2} x^2 y^2\right) f(y) dy, \tag{1.2}$$

where

$$\chi_{\nu, k, m}(x) = x^{-\nu} G_{24}^{21} \left(x \left| \begin{matrix} k-m-\frac{1}{2}, \nu-k+m+\frac{1}{2} \\ \nu, \nu+2m, -2m, 0 \end{matrix} \right. \right). \tag{1.3}$$

He calls $g(x)$ in (1.2) to be the $\chi_{\nu, k, m}$ -transform of $f(x)$ and has proved that this transform is reciprocal like the Hankel transform, i.e. if $g(x)$ is the $\chi_{\nu, k, m}$ -transform of $f(x)$ so is $f(x)$ of $g(x)$. In case $g(x)=f(x)$, i.e. if $f(x)$ is its own $\chi_{\nu, k, m}$ -transform, we may call it self-reciprocal in the $\chi_{\nu, k, m}$ -transform; and may be denoted by $R_{\nu}(k, m)$. It may be noted that on putting $k+m=\frac{1}{2}$; (1.2) reduces to (1.1).

He has established certain kernels in this transform connecting different classes of self-reciprocal functions. Some more new kernels have been evaluated by R. K. Saxena. The object of this paper is to investigate some more new kernels and also to find out some kernels, which are the resultant of two self-reciprocal functions and the resultant of two kernels respectively.

2. Roon Narain⁴ has proved that the necessary and sufficient condition for a function $f(x)$ of $A(\alpha, a)$ to be $R_v(k, m)$ is that it should be of the form

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s/2} \Gamma_x(v/2+s/2+\frac{1}{4}+m\pm m)}{\Gamma(v/2+s/2+\frac{3}{4}+m-k)} \psi(s) x^{-s} ds, \quad (2.1)$$

where $\psi(s)$ is regular and satisfies the equation

$$\psi(s) = \chi(1-s), \quad (2.2)$$

in the strip $a < \sigma < 1-a$. (2.3)

Here we are giving some rules connecting different classes of self-reciprocal functions. We have assumed that $f(x)$ is integrable in $(0, \infty)$ and the integrals involved exist.

3. Rule 1. If $f(x)$ is $R_v(k, m)$ and

$$Q(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma_x(\mu/2+\frac{3}{4}-s/2+n\pm n)}{\Gamma(v/2+s/2+\frac{3}{4}+m-k)} \frac{\Gamma_x(v/2+s/2+\frac{1}{4}+m\pm m)}{\Gamma(\mu/2-s/2+5/4+n-l)} \omega(s) e^{sx} dx, \quad (3.1)$$

$x > 0.$
 $x < 0.$

where c is any positive number and $\omega(s)$ satisfies (5.2), then

$$g(x) = 1/x \int_0^x Q(\log x/y) f(y) dy, \quad (3.2)$$

is $R_{\mu}(l, n)$.

Proof: $f(x)$ is $R_v(k, m)$.

$$\text{Therefore, } f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s/2} \Gamma_x(v/2+s/2+\frac{1}{4}+m\pm m)}{\Gamma(v/2+s/2+\frac{3}{4}+m-k)} \psi(s) x^{-s} ds,$$

and

$$g(x) = 1/x \int_0^x Q(\log x/y) dy = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s/2} \Gamma_x(v/2+s/2+\frac{1}{4}+m\pm m)}{\Gamma(v/2+s/2+\frac{3}{4}+m-k)} \psi(s) x^{-s} ds. \quad (3.3)$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s/2} \Gamma_x(v/2+s/2+\frac{1}{4}+m\pm m)}{\Gamma(v/2+s/2+\frac{3}{4}+m-k)} \psi(s) x^{-s} \int_0^{\infty} e^{(s-1)u} Q(u) du. \quad (3.4)$$

Now, using a form of Mellin's Inversion formula², from (3.1) we have

$$\int_0^{\infty} e^{-\omega u} Q(u) du = \frac{\Gamma_{\times}(\mu/2 - s/2 + \frac{3}{4} + n \pm n) \Gamma_{\times}(v/2 + s/2 + \frac{1}{4} + m \pm m)}{\Gamma(v/2 + s/2 + \frac{3}{4} + m - k) \Gamma(\mu/2 + s/2 + 5/4 + n - l)} \omega(s).$$

Writing $1 - s$ for s , we get

$$\int_0^{\infty} Q(u) e^{(s-1)u} du = \frac{\Gamma_{\times}(\mu/2 + s/2 + \frac{1}{4} + n \pm n) \Gamma_{\times}(v/2 - s/2 + \frac{3}{4} + m \pm m)}{\Gamma(v/2 - s/2 + 5/4 + m - k) \Gamma(\mu/2 + s/2 + \frac{3}{4} + n - l)} \omega(s),$$

because $\omega(s)$ satisfies (3.2).

Therefore,

$$\begin{aligned} g(x) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s/2} \Gamma_{\times}(v/2 + s/2 + \frac{1}{4} + m \pm m) \Gamma_{\times}(\mu/2 + s/2 + \frac{1}{4} + n \pm n)}{\Gamma(v/2 + s/2 + \frac{3}{4} + m - k) \Gamma(v/2 + s/2 + 5/4 + m - k)} \\ &\quad \frac{\Gamma_{\times}(v/2 - s/2 + \frac{3}{4} + m \pm m)}{\Gamma_{\times}(\mu/2 + s/2 + \frac{3}{4} + n - l)} \omega(s) \psi(s) x^{-s} ds \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s/2} \Gamma_{\times}(\mu/2 + s/2 + \frac{1}{4} + n \pm n)}{\Gamma(\mu/2 + s/2 + \frac{3}{4} + n - l)} \phi(s) x^{-s} ds. \end{aligned}$$

$$\text{Where } \phi(s) = \frac{\Gamma_{\times}(v/2 + s/2 + \frac{1}{4} + m \pm m) \Gamma_{\times}(v/2 - s/2 + \frac{3}{4} + m \pm m)}{\Gamma(v/2 + s/2 + \frac{3}{4} + m - k) \Gamma(v/2 - s/2 + 5/4 + m - k)} \omega(s) \psi(s),$$

and $\phi(s)$ satisfies (2.2).

Hence $g(x)$ is $R_{\mu}(l, n)$.

4. *Corollary* : If $\mu = v$ in the above theorem, we get a corollary :

If $f(x)$ is $R_v(k, m)$, the function

$$g(x) = 1/x \int_0^x Q(\log x/y) f(y) dy,$$

is $R_v(k, m)$ provided that

$$\begin{aligned} Q(x) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{sx} \omega(s) ds, & x > 0 \\ &= 0 & x < 0. \end{aligned} \tag{4.1}$$

where $\omega(s)$ satisfies (2.2)

The converse theorem will be like this :

(a) If $f(x)$ is $R_v(k, m)$ and $g(x)$ is $R_\mu(l, n)$ then $Q(x)$ will be given by (3.1).

Another theorem of a similar nature may be given in the form :

(b) If $g(x)$ is $R_\mu(l, m)$ and $Q(x)$ is given by (3.1), then $f(x)$ is $R_v(k, m)$ and will be given by

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s/2} \Gamma_x(\frac{1}{4} + \frac{1}{2}v + \frac{1}{2}s + m \pm m)}{\Gamma(\frac{3}{4} + \frac{1}{2}v + \frac{1}{2}s + m - k)} \psi(s) x^{-s} ds,$$

where $\chi(s)$ satisfies (2.2).

(A) From the corollary, when $\mu = v$ we have :

If $f(x)$ is $R_v(k, m)$ and

$$g(x) = 1/x \int_0^x Q(\log x/y) f(y) dy,$$

is $R_v(k, m)$, then

$$\begin{aligned} Q(x) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \phi(s) e^{xs} dx, & x > 0 \\ &= 0, & x < 0, \end{aligned}$$

where $\phi(s)$ satisfies (2.2)

(B) Another theorem will be : If $g(x)$ is given by the relation

$$g(x) = 1/x \int_0^x Q(\log x/y) f(y) dy$$

is $R_v(k, m)$ and

$$\begin{aligned} Q(x) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \phi(s) e^{xs} ds, & x > 0 \\ &= 0, & x < 0. \end{aligned}$$

then

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s/2} \Gamma_x(\frac{1}{4} + \frac{1}{2}v + \frac{1}{2}s + m \pm m)}{\Gamma(v/2 + s/2 + \frac{3}{4} + m - k)} \psi(s) x^{-s} ds.$$

i.e. $f(x)$ is $R_v(k, m)$

5. Rule II. If $f(x)$ is $R_v(k, m)$ and

$$Q(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^s \Gamma_x(v/2+s/2+\frac{1}{4}+n\pm m) \Gamma_x(\mu/2+s/2+\frac{1}{4}+n\pm n)}{\Gamma(v/2+s/2+\frac{3}{4}+m-k) \Gamma(\mu/2+s/2+\frac{3}{4}+n-l)} \omega(s) e^{sx} ds,$$

$$= O \quad \begin{matrix} x > 0 \\ x < 0 \end{matrix} \quad (5.1)$$

then $g(x) = \int_0^{1/x} Q(\log 1/xy) f(y) dy,$ (5.2)

is $R_\mu(l, n)$ and $\omega(s)$ satisfies (2.2).

The proof is similar to that of theorem 3.

The converse theorem will be :

If $f(x)$ is $R_v(k, m)$ and $g(x)$ is $R_\mu(l, n)$ then

$$Q(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^s \Gamma_x(v/2+s/2+\frac{1}{4}+m\pm m) \Gamma_x(\mu/2+s/2+\frac{1}{4}+n\pm n)}{\Gamma(v/2+s/2+\frac{3}{4}+m-k) \Gamma(\mu/2+s/2+\frac{3}{4}+n-l)} \omega(s) e^{sx} ds,$$

$$= O, \quad \begin{matrix} x > 0, \\ x < 0. \end{matrix}$$

and $\omega(s)$ satisfies (2.2).

The proof is similar to that of (4).

Another theorem is :

If $g(x)$ is $R_\mu(l, m)$ and $Q(x)$ is given by (5.1) then $f(x)$ is $R_v(k, m)$.

6. Roop Narain has given the proposition :

If $f(x)$ is $R_v(k, m)$ and

$$k(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^s \Gamma_x(v/2+s/2+\frac{1}{4}+m\pm m) \Gamma_x(\mu/2+s/2+\frac{1}{4}+n\pm n)}{\Gamma(v/2+s/2+\frac{3}{4}+m-k) \Gamma(\mu/2+s/2+\frac{3}{4}+n-l)} \phi(s) x^{-s} ds,$$
(6.1)

where $\phi(x)$ satisfies (2.2),

then $g(x) = \int_0^\infty f(y) k(xy) dy,$ (6.2)

is $R_{\mu}(l, n)$.

The converse theorem of this will be like this :

Theorem : If $f(x)$ is $R_{\nu}(k, m)$ and $g(x)$ is $R_{\mu}(l, n)$ then $k(x)$ will be given by (6.1).

Multiplying (6.2) by x^{s-1} and integrating between 0 and ∞ we get

$$\int_0^{\infty} x^{s-1} g(x) dx = \int_0^{\infty} f(y) k(xy) dy \int_0^{\infty} x^{s-1} dx.$$

Changing the order of integration, if permissible, we get

$$\int_0^{\infty} x^{s-1} g(x) dx = \int_0^{\infty} f(y) dy \int_0^{\infty} x^{s-1} k(xy) dx,$$

On writing u for xy we get

$$\int_0^{\infty} x^{s-1} g(x) dx = \int_0^{\infty} y^{-s} f(y) dy \int_0^{\infty} u^{s-1} k(u) du.$$

(6.3)

Now $f(x)$ is $R_{\nu}(k, m)$ and $g(x)$ is $R_{\mu}(l, n)$.

Therefore,

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s/2} \Gamma_x(\nu/2 + s/2 + \frac{1}{4} + m \pm m)}{\Gamma(\nu/2 + s/2 + \frac{3}{4} + m - k)} \psi(s) x^{-s} ds,$$

where $\psi(s)$ is regular and satisfies (2.2).

By Mellin's Inversion formula¹,

$$\int_0^{\infty} x^{s-1} f(x) dx = \frac{2^{s/2} \Gamma_x(\nu/2 + s/2 + \frac{1}{4} + m \pm m)}{\Gamma(\nu/2 + s/2 + \frac{3}{4} + m - k)} \psi(s)$$

Writing $1-s$ for s , we get

$$\int_0^{\infty} x^{-s} f(x) dx = \frac{2^{1-s/2} \Gamma_x(\nu/2 - s/2 + \frac{3}{4} + m \pm m)}{\Gamma_x(\nu/2 - s/2 + 5/4 + m - k)} \psi(s)$$

as $\psi(s)$ satisfies (2.2).

And similarly,

$$\int_0^{\infty} x^{s-1} g(x) dx = \frac{2^{s/2} \Gamma_x(\mu/2 + s/2 + \frac{1}{2} + n \pm n)}{\Gamma(\mu/2 + s/2 + \frac{3}{2} + n - l)} \phi_1(s)$$

where $\phi_1(s)$ satisfies (2.2).

Therefore, from (6.3) we have

$$\begin{aligned} \int_0^{\infty} k(u) u^{s-1} du &= \frac{2^{s/2} \Gamma_x(\mu/2 + s/2 + \frac{1}{2} + n \pm n)}{2^{1-s/2} \Gamma_x(v/2 - s/2 + \frac{3}{2} + m \pm m)} \frac{\Gamma(v/2 - s/2 + 5/4 + m - k) \phi_1(s)}{\Gamma(\mu/2 + s/2 + \frac{3}{2} + n - l) \psi(s)}, \\ &= \frac{2^{s/2} \Gamma(v/2 + s/2 + \frac{1}{2} + m \pm m) \Gamma_x(\mu/2 + s/2 + \frac{1}{2} + n \pm n)}{\Gamma(v/2 + s/2 + \frac{3}{2} + m - k) \Gamma(\mu/2 + s/2 + \frac{3}{2} + n - l)} \phi(s) \end{aligned}$$

$$\text{where } \phi(s) = \frac{1}{\sqrt{2}} \frac{\Gamma(v/2 - s/2 + 5/4 + m - k)}{\Gamma_x(v/2 + s/2 + \frac{1}{2} + m \pm m)} \frac{\Gamma(v/2 + s/2 + \frac{3}{2} + m - k)}{\Gamma_x(v/2 - s/2 + \frac{3}{2} + m \pm m)} \frac{\phi_1(s)}{\psi(s)}$$

Hence $\phi(s)$ satisfies (2.2).

By Mellin's Inversion formula¹, we get

$$k(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s/2} \Gamma_x(v/2 + s/2 + \frac{1}{2} + m \pm m) \Gamma_x(\mu/2 + s/2 + \frac{1}{2} + n \pm n)}{\Gamma_x(v/2 + s/2 + \frac{3}{2} + m - k) \Gamma(\mu/2 + s/2 + \frac{3}{2} + n - l)} \phi(s) x^{-s} ds,$$

Thus the theorem is established.

Corollary : If $g(x)$ is $R_{\mu}(l, n)$ and $k(x)$ is given by (6.1),

then $f(x)$ will be given by

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s/2} \Gamma_x(v/2 + s/2 + \frac{1}{2} + m \pm m)}{\Gamma(v/2 + s/2 + \frac{3}{2} + m - k)} \phi(s) x^{-s} ds,$$

where $\phi(s)$ satisfies (2.2) and hence will be $R_v(k, m)$.

He has given another rule in this from ;

8. If $f(x)$ is $R_v(k, m)$ and

$$k(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma_x(v/2 + s/2 + \frac{1}{2} + m \pm m) \Gamma_x(\mu/2 - s/2 + \frac{3}{2} + n \pm n)}{\Gamma(v/2 + s/2 + \frac{3}{2} + m - k) \Gamma(\mu/2 - s/2 + 5/4 + n - l)} \omega(s) x^{-s} ds. \quad (8.1)$$

where $\omega(s)$ satisfies (2.2), then

$$g(x) = 1/x \int_0^{\infty} f(y) k(y/x) dy, \quad (8.2)$$

belongs to $R_{\mu}(l, n)$.

The converse of this theorem will be :

Theorem : If $f(x)$ is $R_{\nu}(k, m)$ and $g(x)$ is $R_{\mu}(l, n)$, then $k(x)$ will be given by (8.1).

The proof follows from the above theorem.

Corollary : If $g(x)$ is $R_{\mu}(l, n)$ and $k(x)$ is given by (8.1), then $f(x)$ will be given by :

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma_{\kappa}(\nu/2 + s/2 + \frac{1}{2} + m \pm m)}{\Gamma(\nu/2 + s/2 + \frac{3}{2} + m - k)} \phi(s) x^{-s} ds,$$

where $\phi(x)$ satisfies (2.2) and hence $f(x)$ is $R_{\nu}(k, m)$.

He has also given another rule, which is immediately obtained from the above rule by putting ν for μ . The rule is :

If $f(x)$ is $R_{\nu}(k, m)$ and $k(x)$ satisfies the relation

$$x k(x) = k(1/x).$$

$$\text{then } g(x) = 1/x \int_0^{\infty} f(y) k(y/x) dy,$$

is also $R_{\nu}(k, m)$.

The converse theorem will be like this :

If $f(x)$ is $R_{\nu}(k, m)$ and

$$g(x) = 1/x \int_0^{\infty} f(y) k(y/x) dy,$$

then $x k(x) = k(1/x)$.

The proof is similar to that of the above theorems.

Another theorem of a similar nature will be :

$$g(x) = 1/x \int_0^{\infty} f(y) k(y/x) dy,$$

and $x k(x) = k(1/x)$,

$$\text{then } f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma_x(\frac{1}{2} + v/2 + s/2 + m \pm m)}{\Gamma(\frac{3}{2} + v/2 + s/2 + m - k)} \phi(s) x^{-s} ds;$$

where $\phi(s)$ satisfies (2.2). i.e. $f(x)$ is $R_v(k, m)$.

9. R. K. Saxena⁵ has given : If $f(x)$ is $R_v(k, m)$ and

$$P(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma_x(v/2 + s/2 + \frac{1}{2} + m \pm m)}{\Gamma(v/2 + s/2 + \frac{3}{2} + m - k)} \frac{\Gamma_x(\mu/2 - s/2 + \frac{3}{2} + n \pm n)}{\Gamma(\mu/2 - s/2 + 5/4 + n - l)} \omega(s) x^{-s} ds. \quad (9.1)$$

where $0 < c < 1$ and $\omega(s)$ satisfies (2.2)

$$\text{then } g(x) = \int_0^{\infty} P(y) f(xy) dy, \quad (9.2)$$

is $R_v(k, m)$. In other words $P(x)$ is a kernel transforming $R_v(k, m)$ into $R_\mu(l, n)$.

The converse theorem may be stated in the following form :

Theorem 11. If $f(x)$ is $R_v(k, m)$ and $g(x)$ is given by the relation

$$g(x) = \int_0^{\infty} P(y) f(xy) dy,$$

then

$$P(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma_x(v/2 + s/2 + \frac{1}{2} + m \pm m)}{\Gamma(v/2 + s/2 + \frac{3}{2} + m - k)} \frac{\Gamma_x(\mu/2 - s/2 + \frac{3}{2} + n \pm n)}{\Gamma(\mu/2 - s/2 + 5/4 + n - l)} \omega(s) x^{-s} ds.$$

where $0 < c < 1$ and $\omega(s)$ satisfies (2.2).

Proof follows by theorem 6.

Another converse theorem of a similar nature may be of this form

$$\text{If } g(x) = \int_0^{\infty} P(y) f(xy) dy.$$

where

$$P(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma_x(\frac{1}{2} + \frac{1}{2}v + \frac{1}{2}s + m \pm m)}{\Gamma(v/2 + s/2 + \frac{3}{4} + m - k)} \frac{\Gamma_x(\mu/2 - s/2 + \frac{3}{4} + n \pm n)}{\Gamma(\mu/2 - s/2 + 5/4 + n - l)} \omega(s) x^{-s} ds.$$

then $f(x)$ is given by the relation

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s/2} \Gamma_x(v/2 + s/2 + \frac{1}{4} + m \pm m)}{\Gamma(v/2 + s/2 + \frac{3}{4} + m - k)} \phi(s) x^{-s} ds,$$

where $\phi(s)$ satisfies (2.2) i.e. $f(x)$ is $R_v(k, m)$.

10. The resultant

$$k(x) = \int_0^\infty f_1(y) f_2(xy) dy,$$

of two function given by

$$f_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s/2} \Gamma_x(v/2 + s/2 + \frac{1}{4} + m \pm m)}{\Gamma(v/2 + s/2 + \frac{3}{4} + m - k)} \phi(x) x^{-s} ds,$$

and

$$f_2(x) = \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{2^{s/2} \Gamma_x(\mu/2 + s/2 + \frac{1}{4} + n \pm n)}{\Gamma(\mu/2 + s/2 + \frac{3}{4} + n - l)} \omega(s) x^{-s} ds,$$

where $c < c' < 1$ and $o < c' < 1$ and $\omega(s)$ and $\phi(s)$ satisfies (2.2)

is a kernel transforming R_v to $R_\mu(l, n)$.

Proof. We know that,

$$k(x) = \int_0^\infty f_2(y) dy \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s/2} \Gamma_x(v/2 + s/2 + \frac{1}{4} + m \pm m)}{\Gamma(v/2 + s/2 + \frac{3}{4} + m - k)} \phi(s) (xy)^{-s} ds$$

Changing the order of integration if permissible we get

$$k(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s/2} \Gamma_x(v/2 + s/2 + \frac{1}{4} + m \pm m)}{\Gamma(v/2 + s/2 + \frac{3}{4} + m - k)} x^{-s} ds \int_0^\infty y^{-s} f_2(y) dy,$$

Applying Mellin's Inversion formula, to the second integral and substituting the value form (10.3) we get

$$k(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma_x(v/2 + s/2 + \frac{1}{4} + m \pm m)}{\Gamma(v/2 + s/2 + \frac{3}{4} + m - k)} \frac{\Gamma_x(\mu/2 - s/2 + \frac{3}{4} + n \pm n)}{\Gamma(\mu/2 - s/2 + 5/4 + n - l)} \phi_1(s) x^{-s} ds,$$

where $\phi_1(s) = \sqrt{2} \phi(s) \omega(s)$, and satisfies (2.2).

Thus $k(x)$ is a kernel transforming $R_\nu(k, m)$ to $R_\mu(l, n)$.

Corollary : On putting $\mu = \nu$ and $k + m = \frac{1}{2}$, we get a particular case due to Dr. Brij Mohan : that is 'the resultant of two R_ν functions is a kernel of the above type'.

Let us consider $f_1(x) = x^{\nu+2m-\frac{1}{2}} e^{-\frac{1}{4}x^2} W_{k, m}(\frac{1}{2}x^2)$,

and $f_2(x) = x^{-\frac{1}{2}}$

The resultant kernel will be given by

$$k(x) = \int_0^\infty (xy)^{\nu+2m-\frac{1}{2}} e^{-\frac{1}{4}x^2y^2} W_{k, m}(\frac{1}{2}x^2y^2) y^{-\frac{1}{2}} dy,$$

is a kernel transforming $R_\nu(k, m)$ into $R_{\frac{1}{2}}(k, m)$ function.

or $k(x) = A/\sqrt{x}$ where A is independent of x and y .

$$11. \text{ The resultant } k(x) = \int_0^\infty P_1(y) P_2(xy) dy, \quad (11.1)$$

of two kernels

$$P_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma_x(\nu/2+s/2+\frac{1}{4}+m\pm m)}{\Gamma(\nu/2+s/2+\frac{3}{4}+m-k)} \frac{\Gamma_x(\mu/2-s/2+\frac{3}{4}+n\pm n)}{\Gamma(\mu/2-s/2+\frac{5}{4}+n-l)} \omega(s) x^{-s} ds, \quad (11.2)$$

and

$$P_2(x) = \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{2^s \Gamma_x(\nu/2+s/2+\frac{1}{4}+m\pm m)}{\Gamma(\nu/2+s/2+\frac{3}{4}+m-k)} \frac{\Gamma_x(\mu'/2+s/2+\frac{1}{4}+n'\pm n')}{\Gamma(\mu'/2+s/2+\frac{3}{4}+n'-l')} \phi(s) x^{-s} ds, \quad (11.3)$$

where $\omega(s)$ and $Q(s)$ satisfy (2.2),

is a kernel transforming $R_\nu(k, m)$ to $R_\mu(n', l')$.

The proof follows the above theorem.

12. *Theorem* : The resultant

$$k(x) = \int_0^\infty P_1(xy) P_2(y) dy,$$

of two kernels of the type

$$P_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c'+i\infty} \frac{2^s \Gamma_x(\frac{1}{4} + \frac{1}{2}v + \frac{1}{2}s + m \pm m) \Gamma_x(\frac{1}{4} + \frac{1}{2}\mu + \frac{1}{2}s + n \pm n)}{\Gamma(\frac{3}{4} + \frac{1}{2}s + \frac{1}{2}v + m - k) \Gamma(\frac{3}{4} + \frac{1}{2}\mu + \frac{1}{2}s + n - l)} \phi(s) x^{-s} ds,$$

and

$$P_2(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c'+i\infty} \frac{2^s \Gamma_x(\frac{1}{4} + \frac{1}{2}v + \frac{1}{2}s + m \pm m) \Gamma_x(\frac{1}{4} + \frac{1}{2}\mu' + \frac{1}{2}s + n' \pm n')}{\Gamma(\frac{3}{4} + \frac{1}{2}s + \frac{1}{2}v + m - k) \Gamma(\frac{3}{4} + \frac{1}{2}\mu' + \frac{1}{2}s + n' - l')} \omega(s) x^{-s} ds,$$

where the above conditions are satisfied, is a kernel transforming $R_{\mu}(n, l)$ to

$$R_{\mu'}(n', l').$$

13. *Theorem.* The resultant

$$k(x) = \int_0^{\infty} P_1(xy) P_2(y) dy,$$

of two kernels of the form.

$$P_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c'+i\infty} \frac{\Gamma_x(v/2 + s/2 + \frac{1}{4} + m \pm m) \Gamma_x(\mu/2 - s/2 + \frac{3}{4} + n \pm n)}{\Gamma(v/2 + s/2 + \frac{3}{4} + m - k) \Gamma(\mu/2 - s/2 + 5/4 + n - l)} \omega(s) x^{-s} ds,$$

and

$$P_2(x) = \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{\Gamma_x(v/2 + \frac{1}{4} + s/2 + m \pm m') \Gamma_x(\mu'/2 - s/2 + \frac{3}{4} + n' \pm n')}{\Gamma(v/2 + s/2 + \frac{3}{4} + m - k) \Gamma(\mu'/2 - s/2 + 5/4 + n' - l')} \phi(s) x^{-s} ds,$$

is a kernel transforming $R_{\mu'}(n', l')$ to $R_{\mu}(n, l)$ provided the above conditions

($0 < c < 1$) and ($0 < c' < 1$) are satisfied, and $\omega(s)$, $\phi(s)$ satisfy (2.2.).

I offer my grateful thanks to Dr. Brij Mohan for his help and guidance in the preparation of this note.

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THE CHEMICAL EXAMINATION OF SPHAERANTHUS INDICUS LINN. PART III : STUDY OF THE NONSAPONIFIABLE MATTER FROM ITS OIL

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ABSTRACT

The nonsaponifiable matter from the oil of *Sphaeranthus indicus* Linn. has been examined in detail and found to contain β -sitosterol, *n*-triacontanol and a hydrocarbon *n*-pentacosane.

In two earlier communications^{1,2} it had been pointed out that the nonsaponifiable matter from the fat of *Sphaeranthus indicus* consisted of sitosterol and another compound which after repeated crystallisation from ethanol melted at 79.5°C and which had been characterised as a higher monohydric alcohol $C_{30}H_{62}O$. It has also been mentioned that in spite of the analytical results nothing definite could be said about the exact composition of this product because alcohols of higher molecular weight are usually mixtures of alcohols with even number of carbon atoms as shown by Chibnall *et. al.*³

Since then a paper on the same subject has appeared⁴ in which the Compound melting point 78-79°C from the nonsaponifiable matter has been referred to as a nonsterol solid. The crude nonsaponifiable matter from the *Sphaeranthus indicus* fat has now been investigated in detail by chromatographing over neutral alumina and eluting with petroleum ether (40-60), ether, chloroform and methanol and their mixtures in different proportions and from the elutes three different products have been isolated and identified. The first product $C_{29}H_{58}O$ melting point 137°C has been found to be identical with β -sitosterol by mixed melting point, characterisation through acetate, benzoate and digitonide and also by the characteristic peaks in the I. R. spectrum. The second compound $C_{30}H_{62}O$ melting point 84-85°C has been identified to be *n*-triacontanol (melting point in literature 86°C) through the preparation and analysis of its acetate, phenylurethane and its oxidation product which has been found to be *n*-triacontanoic acid melting point 91°C. These findings are in agreement with the peaks in the I. R. spectrum of *n*-triacontanol. The third compound is a normal hydrocarbon *n*-pentacosane $C_{25}H_{52}$ melting point 52-53°C.

EXPERIMENTAL

The crude nonsaponifiable matter from the fat of *Sphaeranthus indicus* was chromatographed over alumina and eluted with petroleum ether (40-60), ether, benzene, chloroform methanol and their mixtures in different proportions and the different elutes were examined.

β -Sitosterol—The petroleum ether-benzene fraction gave a product melting point 132°C which after repeated crystallisation from methanol was found to melt at 137°C. It gave all the colour tests for a sterol.

Found C = 83.92%, H = 12.33%, Mol. Wt. = 408, $\alpha_D^{30} = -34.4$

$C_{29}H_{50}O$ requires C = 84.05, H = 12.07, Mol. Wt. = 414, $\alpha_D^{30} = -36.6$

Peaks in the I. R. at 1639, 1460 and 1447 cm^{-1} .

Acetate-melting point 127°C, benzoate melting point 142°C and digitonide melting point 220°C.

n-Triacontanol—From another fraction of the elute a compound melting point 80°C was isolated which after recrystallisation from methanol melted at 84-85°C

Found C = 82.01, H = 14.34, Mol. Wt. = 426

$C_{30}H_{62}O$ requires C = 82.19, H = 14.16, Mol. Wt. 438.

Acetate-melting point 70°C, % Acetyl = 8.02, $C_{30}H_{61}O$ ($\text{CO} \cdot \text{CH}_3$) requires 8.96.

Phenylurethane melting point 94°C (given in literature 96°C)

The alcohol was oxidised with chromic acid and from the mixture an acid melting at 91°C was isolated. The neutralization equivalent of the acid was determined by the method of Tiwari, Srivastava and Sharma⁵ and found to be 445. *n-Triacontanoic acid* melts at 94°C and has the neutralization equivalent 452.

n-pentacosane—Another fraction of the elute on distillation left a white powder melting at 52°C, when recrystallised from ethanol, it melted at 53°C. It gave no tests for any oxygen containing functional group and was found to be saturated in character.

Found C = 85.72, H = 14.28, Mol. Wt. 357,

$C_{25}H_{52}$ requires C = 85.23, H = 14.77, Mol. Wt. 352

The I. R. spectrum shows peaks corresponding to long carbon chain only and for no other structural units.

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CHEMICAL EXAMINATION OF THE BAST FIBRE OF *ABUTILON* *INDICUM*, *SWEET*

By

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ABSTRACT

Chemical examination of the bast fibre of *Abutilon indicum*, *Sweet* was carried out. The incomplete chemical examination carried out earlier, is at considerable variance from the result of the present investigation. The fibre compares well in chemical composition, with other malvaceous fibres of *hibiscus* species.

Abutilon indicum, *Sweet*, (N. O. *Malvaceae*) known as "country mallow", *Kanghi*, *Jhampi*, *Potari*, etc., usually grows wild as small bush, but is also cultivated¹. The leaves, roots and seeds (*Balbij*) are used in medicine^{2,3}. Royle⁴ has not mentioned about the fibre yielding properties of *abutilon* species; but Dodge⁵ has spoken of the fibre as "superior to jute". Watt⁶ has mentioned of little difference from industrial stand point between the fibres of *A. indicum*, *Sweet*, and *A. asiaticum*, *G. Don*. Bal⁷ has reported of fibre from *A. avicennae* as "American jute" and superior to orthodox jute and that the fibre from *A. indicum* is "of less importance used mainly for binding". Duthie⁸ has reported of fibre from *abutilon* species as of much value in Kashmir. It will, thus, be observed that divergent views have been expressed about the suitability of the fibre.

The possibilities of using *A. indicum* fibre (or fibres from other species of *abutilon*) as jute substitute were neither mentioned by Basak and Nodder⁹ nor by Biswas¹⁰. However, in collaboration with Paul¹¹ the spinning and weaving of *A. indicum* fibre (grown in Uttar Pradesh), on the jute mill machineries was later carried out by Biswas when it was found suitable for the production of standard sacking weft, and in admixture with jute, for standard sacking warp. The woven (standard) secking cloth contained 73% of this fibre and 27% of jute.

Cultivation and Harvesting - Seeds collected from the local forests were cultivated, in the rainy season, at the Institute field. Seeds were sown broad cast, (with a heavy seed rate to reduce lateral bracing). The seeds germinated within a few days and the plants, requiring no after care, were in full bloom in about 60 days. The plants were harvested 90 days after germination of the seeds, when bold green pods were formed.

RETTING AND EXTRACTION OF FIBRE

The harvested stems were defoliated and depodded, and steeped with logs, in slow flowing water in a cemented tank. Retting was completed in about 8 days, when the lateral loosening of the fibre bundles took place. The retted stems were washed free from refractions and dried in shade. The fibre was separated by hand. An yield of 10.82% fibre on sun dried, defoliated and depodded stem was obtained.

EXPERIMENTAL

The air-dried fibre was cut mechanically, passed through a 22-mesh screen and received in an air-tight container. The examination of the fibre was carried out after Cross and Bevan¹³, but for the estimation of cellulose some modifications were adopted¹⁴.

The results are given in table 1. Relevant figures due to Betrabet and Navalkar¹² are also given for comparison.

TABLE 1
Examination of *A. indicum* fibre
(on 100g. oven-dry material)

	Authors	Figures due to Betrabet and Navalkar ¹²
Ash	1.41	1.29
Aqueous (cold)	1.51	
extract (hot)	2.13	
1%NaOH solubles	16.89	
Hydrolysis loss	<div style="display: inline-block; vertical-align: middle;"> <div style="display: inline-block; vertical-align: middle;">{</div> <div style="display: inline-block; vertical-align: middle;">alpha beta</div> </div>	<div style="display: inline-block; vertical-align: middle;"> <div style="display: inline-block; vertical-align: middle;">9.94 16.82</div> </div>
Cellulose	70.63	87.48

The amount of cellulose reported by Betrabet and Navalkar¹² is too high and cannot be substantiated even by considering the ecological differences (of Uttar Pradesh and Maharastra) under which the plants were grown.

A portion of the cellulose is resistant to the action of cold (below 20°C.) and concentrated (17.5%) caustic soda solution. The alkali resistant cellulose is termed as alpha-cellulose and is more important from the manufacturer's point of view. The alpha, beta and gamma fractions of cellulose were estimated according to the Tappi method¹⁵. The results are shown in table 2.

TABLE 2
Examination of cellulose of *A. indicum*. Fibre.

Fraction	% on Cellulose	% on Fibre
Alpha cellulose	82.90	58.56
Beta cellulose	0.83	0.62
Gamma cellulose	16.22	11.46

The estimation of cellulose (especially by the Cross and Bevan's method) gives no indication about the pulp yielding properties of the fibrous raw material. To overcome this drawback, Hawley and Fleck¹⁶ have suggested the estimation of hydrolysis number of cellulose. Hydrolysis number was estimated after Howley and Fleck and the results show that the amount of unhydrolysed residue (57.26%) is more or less similar to the alpha cellulose content (58.56%) of the fibre.

TABLE 3

Hydrolysis number of cellulose of *A. indicum* fibre
(on 100g. oven-dry material).

Hydrolysis number of cellulose	18.92
Hydrolysed portion of cellulose on fibre	13.37
Unhydrolysed residue	57.26

Nitrogen was evaluated by Kjeldahl's method, from which the value of protein was calculated. The fatty matter was determined by extracting the fibre with ether, and gums and resins by exhaustive extraction with a mixture of alcohol-benzene (1:2). Lignin was determined by the method of Ost and Wilken- ing¹⁷ as modified by Ritter *et al*¹⁸. Pentosans and furfural was estimated by the method of A. O. A. C.¹⁹ The figures are recorded, alongwith relevant figures due to Betrabet and Navalkar¹², in table 4.

TABLE 4

Non-cellulosic constituents of *A. indicum* fibre
(on 100g. oven-dry material.)

	Authors	Figures due to Betrabet and Navalkar ¹²
Nitrogen	0.25	0.21
Protein (N \times 6.25)	5.31	1.31*
Fat and Wax	0.73	0.497
Gum and resin	1.97	
Lignin	11.81	13.13
Furfural	12.65	
Pentosans	21.62	

Norris and Resch²⁰ have shown that of the total furfural, 23.05% is due to polyuronides. The furfural due to alpha cellulose, according to Sarkar *et al*²¹ is 1.23%. The rest of the furfural is, therefore, due to pentosans. The amount of pentosans present as xylan is calculate from Kröber's table. The results are given in table 5.

* Calculated by the present authors.

TABLE 5
Furfural and pentosans in *A. indicum* fibre
(on 100g. oven-dry material.)

Total furfural		12.65
Furfural from	{ Polyuronides	2.82
	{ Alpha cellulose	0.16
	{ Pentosans	9.57
Pentosans present as xylan		19.67

The pentosan figure does not give a correct indication about the origin of furfural, as in the case of lignified material, the sugar-unit of the cell wall is xylose²². The furfural should accordingly be calculated as xylan. Further, on distillation with a mineral acid, (usually 12% HCl), the widely distributed uronic acids produce furfural and carbon dioxide and whereas the furfural produced is much less than the theoretical, the production of carbon dioxide is quantitative. Carbon dioxide was estimated by the method of Nanji *et al*²³, (by maintaining a boiling time of 3 hours). From the amount of carbon dioxide, the percentage of uronic anhydride was calculated.

In the case of jute, Sarkar *et al*²¹ have shown that, of the total carbon dioxide, only 80% is due to uronic acid. The uronic acid was calculated from the amount of carbon dioxide on the assumption that similar relationships exist also in the case of *A. indicum* fibre. Since hemicelluloses play an important part as the structural units of long vegetable fibres, it was considered necessary to estimate hemicelluloses, for which purpose holocellulose was estimated and prepared after Chattopadhyaya and Sarkar²⁴. The (alcohol-insoluble) hemicelluloses do not, however, represent the entire amount, as a considerable portion is left dissolved in alcohol (50%) used for the precipitation of hemicelluloses from the alkaline solution of holocellulose. The hemicelluloses (alcohol-insoluble) were estimated by following the method of Sarkar *et al*²⁵. The theoretical value of total hemicelluloses was calculated by subtracting the alpha cellulose value from holocellulose figure. The hexosan content was calculated after Das *et al*²⁶. The result of estimation is recorded in table 6.

TABLE 6
Holo- and hemi - celluloses and other related components of *A. indicum* fibre
(on 100g. oven-dry material)

Carbon dioxide	1.18
Polyuronides	3.78
Holocellulose	89.26
Total hemicelluloses (theoretical)	30.70
Hemicelluloses (alcohol-insoluble)	16.86
Hexosan	5.20

A comparison of the analytical data of this and other *malvaceous* fibres (of *hibiscus* species) is made in table 7. It will be observed that this fibre compares well with other fibres of *hibiscus* species in chemical constituents.

TABLE 7

Comparison of *A. indicum* and other malvaceous fibre of *hibiscus* species
(on 100g. oven-dry material)

Constituents	Ash	Fat and wax	Nitrogen	Alkali soluble	Lignin	Furfural	Holocellulose	Alpha cellulose	Total hemicellulose (theoretical-by calculation)	Hem icellulose (alco- hol insoluble	Carbon dioxide	Polyuronides	Xylan	Hexosan
Fibres														
Hibiscus	0.84			19.15	9.81	11.04	80.36	48.99	31.37**		1.50	4.8**		
abelmoschus ²⁷														
H. cannabinus ²⁷	2.00			17.90	12.94	9.95	79.84				1.68	5.38**		
(Mesta)														
H. Cannabinus ²⁸	0.91	0.75	0.38		8.7	10.1	88.2	61.6	26.6**	14.2	1.37	4.54**		
(Mesta)														
H. Cannabinus*	0.73	1.28			7.90			62.94						
(Bimili) ²⁸														
H. esculantus ²⁹	0.93	0.43	0.55	16.11	8.34	12.62	82.02	50.14	31.88	14.84	1.32	4.22	19.72	6.48
H. Sabdariffa ²⁷	0.61			21.05	6.88	9.61	84.07	53.76	30.31**		1.20	3.84		
H. Sabdariffa ²⁸	0.70	0.62	0.21		7.2	10.5	90.0	61.8	28.2**	14.6	1.28	4.09		
var altissima														
Abutilon	1.41	0.73	0.85	16.89	11.81	12.65	89.26	58.56	30.70	16.86	1.18	3.78	19.67	5.30
indicum														

* Represent average. Calculated by the present authors, from figures of two samples of top, middle and bottom portions by the fibre.

** Calculated by the present authors.

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STUDIES IN THE RELEASE OF FIXED POTASSIUM IN CALCAREOUS ALKALINE SOILS OF RAJASTHAN AND FIXATION AND RELEASE OF POTASSIUM UNDER LABORATORY CONDITIONS

By

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ABSTRACT

Soil profiles were collected from different parts of Rajasthan namely Sambhar lake, Chittorgarh, Bhupalsagar, Jaismandand also a soil profile from Soroan (Allahabad). An attempt was made to liberate the potassium occurring in the fixed condition and in the exchangeable state in these calcareous alkaline soils by extracting with normal solutions of ammonium chloride, ammonium acetate, hydrochloric acid and sodium chloride. Also, N/10 hydrochloric acid was used. It was observed that the ammonium ions can release the fixed K ion, but, H and Na ions cannot. The amount of potash released in pounds per acre foot of soil by ammonium ions was very high and varied greatly from profile to profile and also between horizons of the same profile. This variation may be attributed to the type of minerals present and also to the cation exchange capacity of the soil. The unextractable fixed K ions, which were not released by H and Na ions were leferated by ammonium ions. The amount of exchangeable potassium was found to be very small as compared to the amount of fixed potassium.

Gross and net negative charges occurring on the inner and outer surfaces of the soil colloids were determined by extracting the ammonium saturated soils with the normal solutions of the chlorides of sodium and potassium. The gross negative charge represented the amount of ammonium recovered by leaching with N KCl solution and the net negative charge was the amount obtained by subtracting the amount of ammonium recovered by leaching with N Na Cl from the amount of gross negative charge.

The net negative charge represented the amount which occupied the potassium fixing positions.

Effect of the concentration of added K^+ on its fixation and adsorption as influenced by the different replacing ions such as, H, Ca, Mg, Na, Ba and NH_4 , was studied under laboratory conditions. It was observed that as the quantity of potassium added was increased, the amount of fixed K^+ was also increased to a definite limit and then this increase was very small. On the other hand, after this critical point, the amount of exchangeable potassium increased considerably, indicating that the fixation of K^+ in the soil proceeds through the exchange complex, and, as long as the fixing capacity of the soil was not satisfied, the fixation of K ions continued. It was also observed that the ions like Ca, H, Mg, Ba and Na were unable to release the fixed K ions.

Fixation of potassium was studied in the soil presaturated with different ions and the effect of pH on K^+ fixation was also studied. It was observed that the soil having a high pH value and rich in bases like Ca, Mg, Ba and Na was capable of fixing potassium while acid soils were less susceptible to K^+ fixation.

The ability of soils to retain added potassium in a non-exchangeable form has been known for some time, but, only recently it has been demonstrated that soils contain a considerable amount of naturally occurring (native) fixed potassium. Potassium added to soils as fertilizer or from other sources over a period of years cannot be completely accounted for without an assumption that the soils had in some manner converted an appreciable amount of the potash into some difficulty soluble form. The first detailed study of this fixation was that of Volk (43). His results pointed out that (a) drying the soil was an extremely important process in potash fixation, (b) the clay fraction was primarily responsible but the quality

of the clay was also important, since a kaolin clay sample and two laterite soils with large amounts of clay size material did not fix potassium, (c) HCl treatment of a soil decreased its fixing power, whereas treatment with Na_2CO_3 or $\text{Ca}(\text{OH})_2$ increased it; (d) long-continued K^+ - fertilization resulted in decreasing the K^+ - fixing power and increasing the muscovite or muscovite-like component of the clay size fraction and (e) the greater the amount of potash added, the greater was the amount fixed upon drying, but the smaller was the percentage fixed.

The next important contribution was that of Chaminade (11), who related the fixation to the exchange capacity of the soil. He found that maximum potassium fixation resulted if more than 4 per cent of the exchange capacity was occupied by potassium. Chaminade and Drouineau (12) indicated that NH_4^+ and Mg^{++} could also be fixed but that Ca^{++} and Na^+ could not. They suggested that the fixation resulted from a diffusion of the cations into the crystal nuclei of the soil colloids but offered no adequate explanation for the differences among the cations.

The next important contributions were those of Truog and Jones (42), who pointed out the relation between the amount of K fixed and the amount of reduction in exchange capacity of bentonite clay when the clay saturated with K^+ was wetted and dried or when KCl was added to a bentonite sample. Truog and Jones (42) also found that soils in general behaved similarly. They indicated that the exchangeable ions were the seat of the fixation process. Joffe and Kolodny (24) suggested that a part of the exchangeable K ions are converted into a fixed position when one-dimensional lattice contraction takes place as a result of heating.

A clue to the specificity of the fixation reaction for particular cations was provided by the work of Page and Bayer (32). They studied the fixation of cations by the colloids from a sample of Wyoming bentonite (montmorillonite) and from a Miami silt loam soil and connected it to the size of the unhydrated ions. Their results indicated that not only K^+ but NH_4^+ , Rb^+ , Cs^+ and Ba^{++} could also be fixed. They assumed that in the drying process the sheets of montmorillonite contract and the K ions (radius 1.33 Å), having lost their hull of oriented water molecules, are forced into the free hexagonal spaces (radius 1.40 Å) between the Si_2O_5 layers, where they fit tightly. Once in this position, they would be held very firmly by the electrical charge within the crystal lattice. The same reasoning applies to ions having about the same radius such as NH_4^+ (1.21 Å), Ba^{++} (1.31 Å), Rb^+ (1.48 Å) and Cs^+ (1.67 Å). Smaller ions, such as Sr^{++} (1.15 Å), Ca^{++} (0.98 Å), Na^+ (0.98 Å), Mg^{++} (0.71 Å), and Li^+ (0.58 Å) are not fixed because they cannot come close enough to the surrounding O^{--} ions when the mineral is dried with the chloride solutions. Much larger molecules such as $(\text{C}_2\text{H}_5)_2\text{NH}_2^+$, ionic radius (2.33 Å) which cannot enter the spaces, will remain more loosely bound, and upon rehydration they assume their water hull and oscillate freely again and are susceptible for exchange.

Gorbunov (17) postulated that K ions are pressed into the Helmholtz - Stern double layer when the clay mineral of a soil is dried after addition of a solution containing K ions. After the clay is dried, the K ions are no longer exchangeable. The theories of Gorbunov (17) and Page and Bayer (32) are based upon the assumption that K^+ is fixed only when a soil clay mineral is dried with the potassium salt solution. But large amounts of K^+ were fixed in the moist condition, as was found by Sears (37), Chaminade (11). Allaway and Pierre¹, Stanford

and Pierre (38) and Stanford (39). Wiklander (45) found that, Ba ions despite having equal volume to that of K or NH_4 ions were not fixed by micaceous material.

Barshad (4) investigated the basal spacings of montmorillonite saturated with various cations under air-dried conditions and when subsequently immersed in water. A considerable expansion of the mineral lattice occurred when Mg^{++} , Ca^{++} , Ba^{++} , H^+ , Li^+ and Na^+ saturated minerals were immersed in water than when air dried. While with the K^+ , NH_4^+ , and Rb^+ , the lattice remained contracted under both these conditions. These investigations of Barshad⁴ suggested that ions like Ca, Mg, Ba, H, Li and Na cannot be fixed as they are susceptible to rehydration and thereby re-expanding the crystal lattice of the mineral and are easily releasable. While NH_4 , K and Rb ions would be expected to be less available to the displacing cation, and are capable of being fixed in the soil and minerals.

The work of Hendricks, Nelson and Alexander (19) threw some light on the behaviour of K ions. By using controlled vapour hydration of montmorillonite, they found the maximum basal spacing of 12 Å for K^+ saturated material, whereas Ca^{++} , Mg^{++} , Na^+ , and H^+ saturated samples had a value of 16 Å. The 12 Å spacing corresponds to a single layer of water molecules, whereas the 16 Å spacing was equivalent to a double layer. The thermal water loss curves were regarded as indicative of the unhydrated character of the K ions. Jackson and Hellman (25) found that when a K^+ saturated montmorillonite was dried at 30°C it showed a basal spacing of 12.3 Å, whereas the Na^+ and Ca^{++} saturated samples had basal spacings of 14.6 Å and 15.1 Å respectively. The above findings indicate that K ions are not susceptible to hydration when they snugly fit into the lattice "hole" as postulated by Page and Baver (32) and therefore they occupy the space between the sixhexagonally arranged oxygen atoms, they cannot expand the crystal lattice and thereby they cannot be easily released.

According to the statement of Wear and White (46) that when K^+ saturated montmorillonite contracts on drying, the inter layer K ions are pressed into the spaces formed by the superimposition of two voids with hexagonal nets of O ions from each silica layer. But a cation in this position would have fourteen fold coordination and not twelve fold as two ions are in proximity to the spaces. In this position K ions form most stable configuration, as the radius of O ions in fourteen fold coordination with K ions is equal to the radius for these ions (O ions) as calculated by Pauling (33) from wave mechanics ($0.77 = 1.04 \text{ Å}$).

The fore-going statement is a brief review of the views on potassium fixation. A considerable amount of investigation has been carried on the release of the fixed potassium in the soil and minerals. Bartholomew and Janssen (5) in their pot experiments observed that some unavailable K^+ became available in course of time and they postulated this release of K^+ from minerals present in the soil and not from the K^+ which is in the fixed condition. Hoagland and Martin (20) did not obtain results similar to those of Bartholomew and Janssen (5). They concluded that fixed K^+ is not available for the plant growth. Hoagland and Martin (21) found that fixed K^+ cannot be released even by leaching with 0.2 N acid. Blume and Purvis (9) thought that the fixation of K^+ is highly reversible and they stated that potassium in fixed state is released and becomes fixed again in a comparatively short time. Joffe and Kolodny (24) showed that K fixed in soil was practically unavailable to the plant. Breazeale and Magistad (6) found that alternate freezing and thawing failed to release any fixed K^+ . In another experiment they noted that even a prolonged treatment on a steam bath with 0.05

NHCl released no K^+ . Joffe and Levine (26) working with a Dunellen soil profile sample noted only very small quantity of K release from the fixed state even by prolonged treatment with hot HCl.

Stanford and Pierre (38), Barshad (8) and others indicated that the mechanism of ammonium and potassium fixation are alike. Walsh and Murdock (49) observed a considerable quantity of fixed ammonium in several Wisconsin soils in U.S.A. Dhar and *et al* (14) advocated that during the process of photochemical reduction of nitrates into ammonia, a considerable amount of ammonia as NH_4^+ may remain absorbed in tropical soils.

PLAN OF INVESTIGATION

The present investigation was designed to study the following points :—

- (i) Release of fixed and adsorbed K^+ in calcareous alkaline and alkali soil profiles and the effect of various extracting ions upon the release of fixed K^+ .
- (ii) To find out the net negative charge on the soil occupied by the fixed K^+ .
- (iii) Influence of concentrations of potassium salt and effect of different replacing ions on fixation of K^+ in the soil.
- (vi) Fixation of K^+ in the soil saturated with different ions prior to fixation and also the fixation of K^+ as influenced by pH. For studying the fixed and adsorbed potassium, soil profiles designated as A, B, C, D, E, F and G were collected and their location and nature of reaction is described below :—

Profile	Date of collection	Location	Nature of reaction
A	20th June (1958)	Sambhar Lake (JHAPOG) Rajasthan	Alkali pH 10.6 (Surface)
B	Do. ...	Do.	Alkali pH 11.0 (Surface)
C	29th June (1958)	Bhupalsagar (Rajasthan)	Saline and Alkaline. pH 8.0 (Surface)
D	3rd July (1958) ...	Jaismand Lake (Rajasthan)	Saline Alkaline pH 8.0 (Surface)
E	4th July (1958) ...	Chittorgarh (Village Devri) (Rajasthan)	Calcareous Alkaline pH 9.0 (Surface)
F	Do. ...	Chittorgarh (Government I. C. C. C. Farm) Rajasthan	Calcareous Alkaline pH 9.8 (Surface)
G	17th September (1958)	Allahabad (Soroan) Uttar Pradesh	Alkali pH 10.5 (Surface)

EXPERIMENTAL TECHNIQUE

20 g portion of soil passed through 30 mesh sieve were taken, and 250 c. c. of distilled water was added and kept for an hour stirring at intervals. It was filtered through a Whatman No. 42 filter paper placed into a Buchner funnel connected with the filter pump. After filtration with water the sample was washed with 40% alcohol. Soil freed from soluble salts were transferred to a 400 c. c. beaker. 250 c. c. of neutral normal NH_4Ac , NH_4Cl , NaCl , HCl and 0.1 N HCl solutions were added separately to the soil freed from soluble salts. These treated samples were thoroughly stirred and kept overnight. Next day the samples were transferred to Whatman No. 42 filter paper placed into Buchner funnel and leached with the solutions mentioned above in order to collect one litre of extract, 50 c. c. of these extracts were taken and analysed for K^+ .

The value of fixed K^+ was obtained by subtracting the amount of K released by NaCl from the amount of K^+ released by neutral normal ammonium chloride and ammonium acetate. The amount of fixed K^+ released was calculated in pounds per acre foot on the basis of the weight of soil = 4,000,000 LBS/acre foot (U. S. Dept. Agric. Hand book No. 60, 1954). Where the soil samples were leached with 0.1 N HCl and N NaCl they were washed with water till free from chloride and finally with 40% alcohol. They were subsequently leached with neutral N NH_4Cl in order to collect one litre extract. 50 c. c. of these extracts were taken and analysed for K^+ .

In order to find out the electric charge two sets of NH_4^+ saturated samples obtained by leaching with N NH_4Ac were evaporated to dryness, freed from soluble ions by washing with distilled water and 40% alcohol were subsequently leached with neutral N KCl and neutral N NaCl in order to obtain one litre extracts. The amount of NH_4 released was determined in an aliquot portion of extracts by distilling with ignited MgO . The amount of ammonium ion released by Na ions was taken to be equivalent to the exchange capacity of the soil in m. e. per 100 g. soil, and that released by N KCl was taken as gross negative charge on the soil colloids. The difference in the amount of ammonium released by K and Na ions represented the net negative charge occupied by the fixed K ions.

Experiments on potassium fixation in the laboratory were carried out on a normal soil having an exchange capacity 32.30 m. e. %. The soil was passed through 30 mesh sieve. 20 g. soil was weighed (oven dry basis) and different amounts of K^+ as KCl solution of known concentration were added, in order that the amount of K^+ added should range from 1 m. e. to 100 m. e.%. Including blank treatment there were fifteen treatments in total. The moisture was maintained to the moisture holding capacity of the soil at first instance. The soil paste were mixed thoroughly with a glass rod and the treated samples were kept under the fan to get them air dried. Then the samples were moistured with distilled water and transferred to Whatman No. 42 filter paper placed into the Buchner funnel connected to a filter pump and washed with distilled water till the filtrate became free from chloride. Finally the samples were washed with 40% alcohol. The water extracts were collected and raised to a definite volume and analysed for K^+ . This represented the amount of water soluble K^+ in m. e./100 g. soil on oven dry basis. The difference between the amount of K^+ added and the amount of K^+ soluble in water gave the quantity of K^+ adsorbed. These samples were subsequently leached with neutral normal solutions of CaCl_2 , MgCl_2 , BaCl_2 , NaCl and 0.1 N HCl in order to obtain a litre of extract. The amount of K replaced was determined in an aliquot part of the extract. The difference between the amount

of K^+ replaced to the amount adsorbed gave the quantity of K^+ fixed. These quantities were also expressed on the percentage basis with respect to the quantity of K^+ added. The analysis of soil is given along with the analytical results and the analytical results are shown in tables 8 and 9.

In order to determine the effect of H^+ , Ca^{++} , Mg^{++} , Ba^{++} , Na^+ and NH_4^+ ions upon the fixation of K^+ , duplicate samples of the normal soil used in the above experiments saturated with these ions were obtained by treating with neutral normal chloride solutions of these ions and removing the excess of salt by washing with distilled water. These H^+ , Ca^{++} , Mg^{++} , Ba^{++} , Na^+ and NH_4^+ soils thus obtained were subsequently treated with 100 c. c. of neutral normal KCl solution. In order to find out the effect of H ion concentration on K^+ fixation a normal solution of alcoholic KOH were added to another set of soil samples saturated with the above mentioned ions. Treated samples were kept over night, next day these samples were heated to dryness on a water-bath. Then the samples were moistened again with distilled water and transferred to Whatman No. 42 filter paper placed in the Buchner funnel connected with filter pump and leached with neutral N KCl and N KOH alcoholic, and then with distilled water till free from soluble salts. Then the samples were leached with N NH_4Cl . The amount of K^+ released was determined in the NH_4Cl extract thus obtained. The difference in the amount of K found in the leachate and the exchange capacity gave the quantity of K^+ fixed. The results are shown in table 10.

In all these experiments potassium was estimated by the sodium cobaltinitrite method. Methods quoted by Write and Piper (50) were used for soil analysis.

TABLE 1

The Amount of Potassium released and cation exchange capacity of different soil profiles in m. e. %.

Extracting Solutions used.	DEPTH IN INCHES							
	0. to 6	6 to 18	18 to 30	30 to 42	42 to 54	54 to 66	66 to 78	
<i>Profile No. A.</i>								
C. E. C.	...	18.78	23.05	27.24	38.38	36.84	40.46	42.22
NH_4Ac	...	87.48	91.89	116.47	138.05	131.95	183.44	190.50
NH_4Cl	...	86.66	90.80	116.24	136.52	130.92	181.82	190.00
0.1 N HCl	...	6.76	7.09	8.04	9.77	10.30	10.32	10.59
N HCl	...	1.52	2.07	2.32	2.75	2.57	2.88	3.50
N NaCl	...	1.07	1.44	2.45	2.65	2.52	2.59	2.74
<i>Profile No. B.</i>								
C. E. C.	...	16.81	25.57	30.55	14.79	13.38	14.63	15.20
NH_4Ac	...	93.86	40.09	159.33	103.67	69.75	42.03	46.25
NH_4Cl	...	93.07	138.66	158.82	98.04	69.75	40.99	46.03
0.1 N HCl	...	4.24	5.75	6.62	2.26	2.79	2.48	2.63
N HCl	...	1.58	1.63	1.85	0.39	0.36	0.37	0.38
N NaCl	...	1.37	1.78	2.44	0.65	0.52	0.43	0.47

Table I (Concd.)

<i>Profile No. C.</i>								
C. E. C.	...	18.47	22.29	22.73	18.48	14.79	12.84	...
NH ₄ Ac	...	56.27	56.85	62.40	65.23	126.36	128.66	...
NH ₄ Cl	...	55.76	56.28	60.93	64.49	126.05	126.36	...
0.1 N HCl	...	1.45	1.39	2.26	2.71	3.25	3.40	...
N HCl	...	0.22	0.46	0.52	0.62	0.65	0.66	...
N NaCl	...	0.74	0.38	1.06	1.20	1.42	1.52	...
<i>Profile No. D.</i>								
C. E. C.	...	19.23	20.54	24.20	19.85	15.60	15.38	...
NH ₄ Ac	...	32.33	32.50	84.34	92.44	109.82	112.36	...
NH ₄ Cl	...	30.36	30.75	83.86	90.92	108.90	111.49	...
0.1 N HCl	...	1.30	1.74	2.42	2.61	2.04	2.22	...
N HCl	...	0.22	0.27	0.28	0.32	0.36	0.40	...
N NaCl	...	0.32	0.42	0.63	0.67	0.81	0.85	...
<i>Profile No. E.</i>								
C. E. C.	...	20.63	18.36	22.79	24.58	15.60	16.27	...
NH ₄ Ac	...	34.60	42.62	86.07	132.63	124.32	127.36	...
NH ₄ Cl	...	33.92	42.05	85.63	130.98	134.02	127.33	...
0.1 N HCl	...	3.28	2.93	2.09	3.41	2.09	2.98	...
N HCl	...	0.22	0.24	0.53	0.84	0.46	0.42	...
N NaCl	...	0.58	0.75	0.95	1.52	1.40	1.47	...
<i>Profile No. F.</i>								
C. E. C.	...	21.20	22.45	30.08	34.75	26.80	24.25	...
NH ₄ Ac	...	26.37	38.40	66.22	79.48	33.62	28.05	...
NH ₄ Cl	...	26.12	37.59	66.08	78.98	33.44	27.69	...
0.1 N HCl	...	1.88	2.20	2.72	3.46	2.04	1.98	...
N HCl	...	0.22	0.26	0.40	0.52	0.29	0.25	...
N NaCl	...	0.54	0.59	0.89	0.92	0.52	0.46	...
<i>Profile No. G.</i>								
C. E. C.	...	8.36	9.69	13.88	15.66	13.66	9.98	8.98
NH ₄ Ac	...	18.26	19.81	27.24	39.79	42.34	24.62	28.32
NH ₄ Cl	...	17.94	19.20	26.75	39.09	0.45	24.06	27.88
0.1 N HCl	...	2.94	2.85	3.06	3.24	3.68	2.02	2.35
N HCl	...	0.38	0.39	0.57	0.32	0.94	0.67	0.75
N NaCl	...	0.24	0.21	0.36	0.78	0.82	0.53	0.63

The percentage of Sesquioxide released by extracting with N HCl in different Soil profiles.

A	...	2.35	2.57	2.69	3.32	3.79	3.65	3.77
B	...	1.74	2.07	2.55	1.32	1.35	1.23	1.32
C	...	2.45	2.72	3.22	3.48	3.07	3.29	...
D	...	1.79	2.05	2.48	2.79	2.30	2.39	...
E	...	2.68	2.56	2.46	3.12	3.20	3.40	...
F	...	1.55	2.36	2.79	2.98	2.28	2.33	...
G	...	1.26	1.57	1.94	1.22	2.48	1.58	1.66

TABLE 2
Exchangeable K⁺ in calcareous alkaline soil profiles (1 lb./acre foot soil).

		DEPTH IN INCHES						
		0 to 6	6 to 18	18 to 30	30 to 42	42 to 54	54 to 66	66 to 78
<i>Profile A</i>	...	1674.0	3035.0	3833.0	4145.0	3941.0	4052.0	4286.0
<i>Profile B</i>	...	2184.0	2784.0	3817.0	1016.0	813.4	672.6	735.2
<i>Profile C</i>	...	1157.0	1377.0	1652.0	1877.0	2221.0	2378.0	...
<i>Profile D</i>	...	500.4	656.09	995.04	1048.02	1267.0	1267.0	...
<i>Profile E</i>	...	907.2	1173.0	1486.0	2378.0	2190.0	2293.0	...
<i>Profile F</i>	...	844.7	923.0	1329.0	1439.0	813.4	719.6	...
<i>Profile G</i>	...	375.4	320.5	563.1	1220.0	1282.0	829.0	992.2

TABLE 3
Fixed K⁺ in calareious alkaline soil profiles.
(m. c /100 g soil, oven dry).

		DEPTH IN INCHES						
		0 to 6	6 to 18	18 to 30	30 to 42	42 to 54	54 to 66	66 to 78
<i>Profile A</i>								
N NH ₄ Ac	...	86.44	89.92	114.02	135.40	129.44	180.85	187.76
N NH ₄ Cl	...	85.62	88.83	113.79	133.87	128.41	179.23	187.25
<i>Profile B</i>								
N NH ₄ Ac	...	92.49	138.31	156.89	103.02	69.23	41.60	45.78
N NH ₄ Cl	...	91.70	136.88	156.38	97.39	68.71	40.56	45.56
<i>Profile C</i>								
N NH ₄ Ac	...	55.53	55.97	61.34	64.03	124.94	127.14	...
N NH ₄ Cl	...	55.02	55.40	59.87	63.29	124.63	124.84	...
<i>Profile D</i>								
N NH ₄ Ac	...	32.01	32.08	83.61	91.77	109.08	111.57	...
N NH ₄ Cl	...	30.54	30.33	83.23	90.25	108.09	110.64	...
<i>Profile E</i>								
N NH ₄ Ac	...	34.02	41.87	85.42	131.11	123.92	126.39	...
N NH ₄ Cl	...	33.34	41.30	84.68	120.46	122.92	125.86	...
<i>Profile F</i>								
N NH ₄ Ac	...	26.37	38.40	66.22	79.48	33.62	28.05	...
N NH ₄ Cl	...	26.12	37.59	66.08	78.98	33.44	27.69	...
<i>Profile G</i>								
N NH ₄ Ac	...	18.02	16.60	26.88	39.01	41.52	24.09	27.70
N NH ₄ Cl	...	17.70	18.99	25.39	33.31	39.63	25.53	27.26

TABLE 4
Fixed K⁺ in calcareous alkaline soil profiles.
(1 lb /acre/foot of soil).

		D E P T H I N I N C H E S						
		0 to 6	6 to 18	18 to 30	30 to 42	42 to 54	54 to 66	66 to 78
<i>Profile A.</i>								
N NH ₄ Ac	...	135200	170900	178300	211700	202600	282300	298000
N NH ₄ Cl	...	134000	154600	177900	211500	200800	280300	293700
<i>Profile B.</i>								
N NH ₄ Ac	...	144600	216600	245400	161100	108100	91030	93250
N NH ₄ Cl	...	143400	214100	244600	152400	107500	63450	71270
<i>Profile C.</i>								
N NH ₄ Ac	...	86860	87540	95960	100200	195500	198800	...
N NH ₄ Cl	...	86080	86660	93650	99000	194900	195200	...
<i>Profile D.</i>								
N NH ₄ Ac	...	50060	50180	130800	143500	170500	174400	...
N NH ₄ Cl	...	47770	47430	130200	139600	169000	173000	...
<i>Profile E.</i>								
N NH ₄ Ac	...	53220	65490	133700	204600	193900	197700	...
N NH ₄ Cl	...	92140	64620	132400	202600	192400	196900	...
<i>Profile F.</i>								
N NH ₄ Ac	...	40400	59140	103900	122800	65070	71610	...
N NH ₄ Cl	...	40000	57880	102100	114300	63450	71270	...
<i>Profile G.</i>								
N NH ₄ Ac	...	28180	30660	42040	61020	64940	37680	43330
N NH ₄ Cl	...	27690	29710	39710	59940	61980	36810	42590

TABLE 5

Release of fixed K^+ not released by H and Na ions by leaching with neutral N NH_4Cl in m. e./100 g soil (oven dry).

		DEPTH IN INCHES						
		0 to 6	6 to 18	18 to 30	30 to 42	42 to 54	54 to 66	66 to 78
<i>Profile A.</i>								
0.1 N HCl	...	85.12	88.52	113.79	133.40	128.34	179.02	186.35
N NaCl	...	85.29	88.80	112.20	133.78	128.48	179.06	187.12
<i>Profile B.</i>								
0.1 N HCl	...	92.12	134.00	156.12	101.62	69.07	40.62	45.39
N NaCl	...	91.74	137.04	156.36	101.04	68.66	40.20	45.32
<i>Profile C.</i>								
0.1 N HCl	..	55.02	55.44	60.14	63.37	125.08	125.52	...
N NaCl	.	55.04	55.28	60.08	63.28	124.45	125.04	...
<i>Profile D.</i>								
0.1 N HCl	...	30.57	30.45	83.37	90.60	108.54	108.16	...
N NaCl	...	31.04	30.04	83.15	91.22	107.76	110.50	...
<i>Profile E.</i>								
0.1 N HCl	...	33.59	41.78	85.02	130.14	123.18	126.59	...
N NaCl	...	32.97	40.72	84.34	124.29	122.19	125.40	...
<i>Profile F.</i>								
0.1 N HCl	...	25.79	37.22	65.26	78.40	32.06	27.32	...
N NaCl	...	26.20	37.60	66.04	78.49	32.79	27.66	...
<i>Profile G.</i>								
0.1 N HCl	...	17.29	18.66	25.98	38.28	39.13	23.05	25.16
N NaCl	...	17.22	18.90	26.25	38.30	34.24	23.41	27.20

TABLE 6

Release of fixed and absorbed NH_4^+ in calcareous alkaline soil profiles saturated with $\text{NH}_4\text{-Ac}$ in m. e./100 g soil oven dry. determination of gross and net negative charges.

		DEPTH IN INCHES						
		0 to 6	6 to 18	18 to 30	30 to 42	42 to 54	54 to 66	66 to 78
<i>Profile A.</i>								
Gross	...	104.69	113.26	142.03	173.79	165.98	220.87	229.36
Exchange Complex		18.78	23.05	27.24	38.38	36.84	40.46	42.22
Net	..	85.91	90.21	114.79	145.48	129.14	180.41	187.14
<i>Profile B.</i>								
Gross	...	108.82	164.13	187.27	117.66	83.08	56.24	60.79
Exchange Complex		16.81	25.57	30.55	14.79	13.88	14.63	15.20
Net	...	92.01	138.56	156.72	102.87	69.20	41.61	45.59
<i>Profile C.</i>								
Gross	...	74.32	78.08	84.21	82.36	139.56	140.27	...
Exchange Complex		18.47	22.29	22.73	18.48	14.79	12.84	...
Net	...	55.85	55.79	61.48	63.88	124.77	127.43	...
<i>Profile D.</i>								
Gross	...	50.79	53.02	108.22	106.49	124.63	127.20	...
Exchange Complex...		19.23	20.54	24.20	14.85	15.60	15.88	...
Net	...	31.56	32.48	84.02	91.64	109.08	111.32	...
<i>Profile E.</i>								
Gross	...	54.63	60.09	108.24	155.70	139.52	142.43	...
Exchange Complex		20.63	18.36	22.79	22.58	15.60	16.27	...
Net	...	34.00	41.73	85.45	113.12	123.92	126.21	...
<i>Profile F.</i>								
Gross	...	46.88	60.95	96.47	114.32	62.60	52.46	...
Exchange Complex		21.20	22.45	30.08	34.75	26.80	24.25	...
Net	...	25.68	37.50	66.39	79.57	35.80	28.21	...
<i>Profile G.</i>								
Gross	...	26.54	30.82	40.02	55.70	54.57	34.92	37.08
Exchange Complex		8.38	10.24	12.79	16.08	14.05	10.58	9.63
Net	...	18.16	20.58	27.23	39.62	40.52	24.34	27.45

Analysis of the soil for K⁺ fixation under laboratory conditions.

<i>Total composition</i>		<i>Mechanical composition</i>	
Moisture	... 1.726%	Coarse sand	... 18.4%
Loss on ignition	... 4.862%	Fine sand	... 32.5%
HCl insoluble	... 72.486%	Silt	... 26.9%
R ₂ O ₃	... 16.660%	Clay	... 17.4%
Fe ₂ O ₃	... 5.443%	Loss due to solution	... 4.4%
CaO	... 2.356%		
MgO	... 0.875%		
K ₂ O	... 0.317%		
P ₂ O ₅	... 0.225%		
Organic carbon	... 0.784%		
Total nitrogen	... 0.071%		
C : N ratio	... 10.9		

Physical constants

C. E. C.	... 32.3 m. e. %
Exchangeable calcium	... 12.2 m. e. %
Exchangeable magnesium	4.7 m. e. %
Exchangeable K.	.. 0.2 m. e. %
Base saturation	... 53.3 %
Water holding capacity...	40.5 %
Permeability	... 16.5 cc/hr.
Total soluble salts	... 0.847 %
pH	... 7.20

N.B.—The soil having the above composition was passed through a 30 mesh sieve and used for fixation of K⁺ under laboratory conditions.

TABLE 7

Experiments on K. fixation under laboratory conditions. Influence of concentration of potassium salt (KCl) and different replacing ions on fixation of K in soil.

Treat-ments.	Amount of K ⁺ added in m.e./ 100g Soil	Amount of K ⁺ water soluble		Amount of K ⁺ adsorbed.	
		In m.e./ 100g Soil	Percentage of added.	In m.e./ 100g Soil	Percentage of added.
1	0.00	0.00	0.00	0.00	0.00
2	1.00	0.00	0.00	1.00	100.00
3	2.50	0.05	2.00	2.45	98.00
4	5.00	0.25	5.00	4.75	95.00
5	10.00	0.84	8.40	9.17	91.60
6	15.00	2.26	15.05	12.74	84.95
7	20.00	3.50	17.50	16.50	82.50
8	25.00	4.58	18.32	20.42	81.68
9	30.00	6.95	23.14	23.05	76.86
10	35.00	8.88	25.30	26.12	74.70
11	40.00	10.50	26.26	29.12	73.75
12	45.00	12.62	28.02	32.38	71.98
13	50.00	14.74	29.98	35.26	76.52
14	75.00	22.95	30.22	52.05	69.78
15	100.00	33.63	33.63	66.37	66.37

TABLE 8

The Amount of Exchangeable and Non-exchangeable (Fixed) Potassium with replacing ions H as 0.1 N HCl and as normal solutions of Chlorides of Calcium, Magnesium, Barium, Sodium and Ammonium.

Treat-ments No.	Amount of K added in m.e. %	Amount of K exchangeable with replacing ions.					
		H	Ca	Mg	Ba	Na	NH ₄
1	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2	1.00	0.00	0.00	0.00	0.00	0.00	1.00
3	2.50	0.03	0.03	0.05	0.06	0.07	2.45
4	5.00	0.13	0.08	0.10	0.14	0.14	4.75
5	10.00	0.28	0.20	0.44	0.50	0.49	9.13
6	15.00	0.48	0.56	0.68	0.82	0.77	12.67
7	20.00	0.67	0.88	0.92	1.09	1.05	16.18
8	25.00	1.16	1.20	1.35	1.42	1.42	19.97
9	30.00	1.44	1.75	1.82	2.01	1.87	22.26
10	35.00	1.77	2.20	2.27	2.43	2.46	25.24
11	40.00	2.28	2.52	2.66	2.79	2.88	28.04
12	45.00	2.75	3.04	3.08	4.12	3.38	31.03
13	50.00	3.26	3.56	3.72	4.74	3.92	33.63
14	75.00	18.81	19.05	20.14	21.06	19.24	49.47
15	100.00	31.16	31.89	32.02	34.07	32.39	63.05

TABLE 9

Amount of K Non-exchangeable (Fixed with replacing ions.)

1	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2	1.00	1.00	1.00	1.00	1.00	1.00	0.00
3	2.50	2.42	2.42	2.40	2.39	2.33	0.00
4	5.00	4.12	4.67	4.65	4.61	4.61	0.00
5	10.00	8.89	8.87	8.73	8.67	8.68	0.04
6	15.00	12.26	12.18	12.06	11.92	11.97	0.07
7	20.00	15.83	15.62	15.58	15.41	15.45	0.32
8	25.00	19.26	19.22	19.07	19.00	19.00	0.45
9	30.00	21.61	21.30	21.23	21.04	21.18	0.79
10	35.00	24.35	23.92	23.85	23.69	23.66	0.88
11	40.00	26.84	26.60	28.46	26.33	26.24	1.08
12	45.00	29.63	29.34	29.30	28.26	29.00	1.35
13	50.00	32.00	32.60	31.54	30.52	31.34	1.63
14	75.00	33.24	33.00	31.81	30.99	32.81	2.58
15	100.00	35.21	34.48	34.35	32.30	33.98	3.32

TABLE 10

Fixation of K^+ in Soil Saturated with Different ions Prior to Fixation and
Fixation Influenced by pH
(m.e./100g Soil)

Saturating ions prior to fixation	FORMS OF K-APPLIED	
	N KCl (neutral) m.e. %	N KOH (alcoholic) m.e. %
H^+		
K recovered	34.33	60.24
K fixed	2.03	28.21
Ca^{++}		
K recovered	48.46	64.28
K fixed	16.16	31.98
Mg^{++}		
K recovered	48.08	62.79
K fixed	15.78	30.49
Ba^{++}		
K recovered	50.27	65.26
K fixed	17.97	32.96
Na^+		
K recovered	46.72	60.28
K fixed	14.42	27.98
NH_4^+		
K recovered	52.04	66.57
K fixed	19.74	34.27

DISCUSSION

The foregoing results (table 1) show that the fixed K^+ is present in all calcareous alkaline soil profiles in all the horizons, which can easily be released by leaching with neutral normal ammonium acetate and ammonium chloride solutions and not by N NaCl or 0.1 N HCl or even by normal HCl. Hoagland and Martin (20), Joffe and Levine (26) could not obtain any release of fixed K^+ by prolonged treatment with hot 10 N HCl. It indicates that H and Na ions are not able to enter the interior of the crystal lattice of the colloidal matter or they are not able to pass into the lattice "hole" as pointed out by Page and Haver (32) while NH_4 ion can pass through the lattice "hole" and replace the fixed K ions. It is evident from the table 1 that the release of fixed K^+ by NH_4 ions increases with depth in the profile A, C, and D. But in the case of profile B, E and F the maximum release is observed in the 4th horizon (30"—42") and in the 5th horizon of profile G, then it decreases. This decrease from the 4th horizon downwards is greater in profile B and F, but smaller in profile E and G. Although the quantity of K^+ released by leaching with N. NaCl and 0.1 NHCl and N HCl is very small, but the same trend of increase or decrease in the K^+ replacement is noted (Table 1).

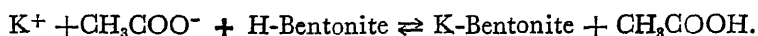
There is no release of R_2O_3 when extracted by N NH_4Cl , N NH_4Ac and 0.1 N HCl and N NaCl; but there is considerable release of R_2O_3 in the case of N HCl extract. There seems to be no direct relationship between the release of R_2O_3 and adsorbed K^+ . But it is definite that the greater release of K^+ with N HCl and no release by 0.1 N HCl confirms some break down of colloidal complex and there by some release of K^+ (Table 1). Joffe and Levine (26) reported small but steady increase in K^+ release with increasing acid concentration. Joffe and Levine (26) used normal Na_2CO_3 in releasing the fixed K ions and noticed no release of K^+ and R_2O_3 . This further points out that the fixation of K^+ in the alkaline conditions in the soil is more pronounced than in the acidic conditions. And alkali soils are more susceptible to K^+ fixation than the acidic soils.

There seems to be no direct relationship between the release of K^+ and the cation exchange capacity of the soil. It is evident from Table 1 that in certain profile there is decrease in the value of cation exchange capacity with depth, but the amount of K^+ released is greater. This shows that some of the positions in the exchange complex must have been occupied by the fixed K ions which have lowered the exchange capacity of the soil.

Levine and Joffe (28) working on K^+ fixation with a sample of Montalto hydrogen colloid observed a reduction in the exchange capacity when K ions were fixed under laboratory conditions. Their results show that with increasing K^+ fixation the decrease in the exchange capacity becomes larger. Similar observations were recorded by Truog and Jones (42). It appears that when all the fixing capacity of a soil is satisfied, some of the exchange positions are transformed into forms which are responsible for the fixation of ions.

The statement of the fact that potassium fixation follows reduction in the exchange capacity of a soil and if the reduction in the exchange capacity over fixation is an additive property a time would come when a good soil might deteriorate to a great extent by losing its property of base exchange phenomena. Potassium being an alkali metal would bring an alkali soil to a still worse condition when present in the fixed condition. The presence of fixed potassium in alkali soil may be one of the reasons of its deterioration and its becoming unfit for cultivation.

The results in table 1 show that the amount of adsorbed K^+ obtained by extracting the soil with normal ammonium acetate is slightly higher than that obtained by extracting with ammonium chloride. Hoagland and Martin (21) and Chaminade (11) reported that the nature of anions exerts no influence on K^+ fixation. Wiegner (47) used anions of very weak acid. He studied the adsorption of CH_3COOK on H bentonite and obtained greater adsorption of K^+ as compared with chloride ions, and he explained this as follows



Similar results were obtained by Joffe and Levine (27).

The values of exchangeable K^+ are given in Table 2. These are the amounts of K^+ released in pounds per acre foot of soil by extracting the soil with normal NaCl solution. By subtracting the values of exchangeable K^+ (the amount of K released by N NaCl) (Table 1) from the values of adsorbed K^+ , the values of fixed K^+ were obtained (Table 3). The values of fixed K for ammonium acetate and ammonium chloride are recorded separately in m.e.%. These values of fixed K were calculated in pounds per acre foot of soil, which are recorded in Table 4. It can be seen from Tables 2 and 4 that there is a great variation in the amount of fixed and exchangeable K^+ obtained between the profiles and between the horizons of each profile. The variation in the amount of fixed K^+ found between different profiles may be due to the type of minerals present in them. The most important type of the clay minerals responsible for the fixation of K^+ are vermiculite and illite and under certain conditions montmorillonite, and the soil containing such minerals are responsible for K^+ fixation. Ahoe (2), Barshad (7, 8) Hanway Scott and Stanford (22), Martin and Overstreet and Hoagland (30), Stanford (41), Troug and Jones (42), Volk, Mitra and Dharmprakash (43) observed that these minerals are responsible for K^+ fixation. The difference in the amount of fixed K^+ found in different profiles may be due to the leaching of clay from one horizon and getting accumulated into another horizon. From Table 4 it appears that the amount of fixed K^+ in the profile G is much less than the amount in the other six profiles. Profile G was collected from Soraon (Allahabad U. P.). It seems that the proportion of K^+ fixing minerals present in this profile is less than in the other six profiles which were collected from Rajasthan. Dyer (15), Frear and Erb (16), Page and Williams (34), concluded from the potassium crop balance in long term field experiments that much of the potassium added as manures was fixed by the soil. Volk (43) reported that 2115 Lbs. of K_2O per acre are fixed by Hagers town silt loam, receiving 3188 Lbs of K_2O per acre in 50 years. Blume and Purvis (9), Harris (23) studied the fixation of K and the effect of lime and magnesia upon K^+ fixation. Barshad (10) defined the naturally fixed K^+ as native fixed potassium.

The amount of K^+ released by subsequently leaching the soil samples already extracted with 0.1 N HCl and N NaCl with normal ammonium chloride are shown in Table 5. The results obtained show that there is a considerable release of the K^+ which was not released by extracting with 0.1 N HCl and N NaCl. And the amount of K^+ released is just equivalent to the amount released by directly leaching with N NH_4 Cl, when the amount of K^+ released by 0.1N HCl and N NaCl and the amount of K^+ released subsequently by leaching with N NH_4 Cl are added together.

Results shown in Table 6 represents the gross and net negative charges and the negative charge on the exchange complex i.e. the exchange capacity. Gross

negative charge is the amount of NH_4^+ released by extracting the NH_4Cl saturated soil with normal KCl , and the negative charge at the exchange complex is the amount of NH_4 replaced by leaching with N NaCl . The net negative charge is the difference of the gross negative charge and the negative charge of the exchange complex (cation exchange capacity). The value of net negative charge is just equivalent to the amount of K^+ released with little variations. This shows that the fixed K^+ and NH_4^+ can be replaced by each other and not by any other ions. It can be seen in Table 6 that there is a considerable variation between the net negative charge and the gross negative charge from profile to profile indicating that different soils have different net negative charge and thereby their fixing capacity for K^+ and NH_4^+ differs from one soil to another.

Influence of concentration of added K^+ as KCl , on fixation and adsorption of K^+ and the effect of different replacing ions on fixation of K^+ was also studied. The results obtained are shown in Tables 8 and 9, in which the effect of H^+ , Ca^+ , Mg^{++} , Ba^{++} , Na^+ and NH_4^+ ions on the fixation of K ion has been studied. It can be seen in these tables that as the amount of added K^+ is increased, the amount of adsorbed K^+ also increases, but the percentage decreases as the amount of added K^+ increases (Table 7). The amount adsorbed K^+ was plotted against the amount of K^+ added (Fig. 1). It is evident in (Fig. 1) that the amount of K^+

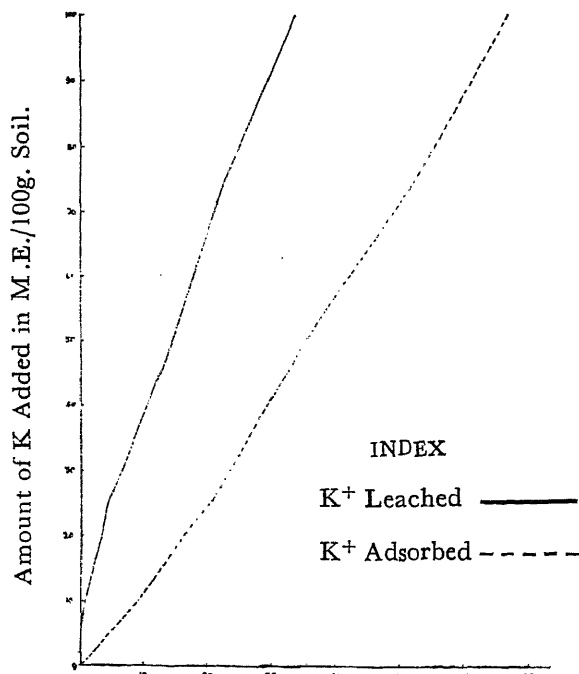


Fig. 1. Amount of K^+ Leached and Adsorbed in M.E./100g. Soil.

adsorbed increases almost linearly with the amount of K^+ added. As the amount of K^+ added increases the amount of water soluble K^+ increases and its percentage also increases simultaneously (Table 7). The amount of water soluble K^+ was also plotted against the amount of K^+ added (fig. 1) with the ordinate and abscissa of equal scales.

The decrease in the percentage of adsorbed K^+ may be explained on the basis that with the increase in the application of K^+ the fixing capacity of the soil is satisfied i.e. when a small quantity of K^+ is applied the fixing capacity and the exchange capacity of the soil is not a limiting factor and a large proportion of the added K^+ enters into the exchange complex and a large proportion is adsorbed. On the other hand when higher doses of K^+ are applied the fixing capacity and the exchange capacity are satisfied and although the absolute quantity of adsorption increases it does so at a decreasing rate. The adsorbed K^+ ions on the exchange complex slowly migrate and occupy the fixing positions and thereby lower the exchangeable K^+ . This shows that the fixation of ions in the soil proceeds through the exchange complex as suggested by Joffe and Levine (28). Bartholomew and Jansen (5), Chaminade and Drouineau (12), and Schachtschabel (4) stated that the non-exchangeable, the exchangeable and the ionic forms of K^+ in the soil solution are in equilibrium.

The amount of K^+ which remained fixed in the soil against extraction with H^+ , Ca^{++} , Mg^{++} , Ba^{++} , Na^+ and NH_4^+ (Table 9) was plotted against the amount of K^+ added per 100 g of oven dry soil (Fig. 2). Graph was plotted with the ordinate

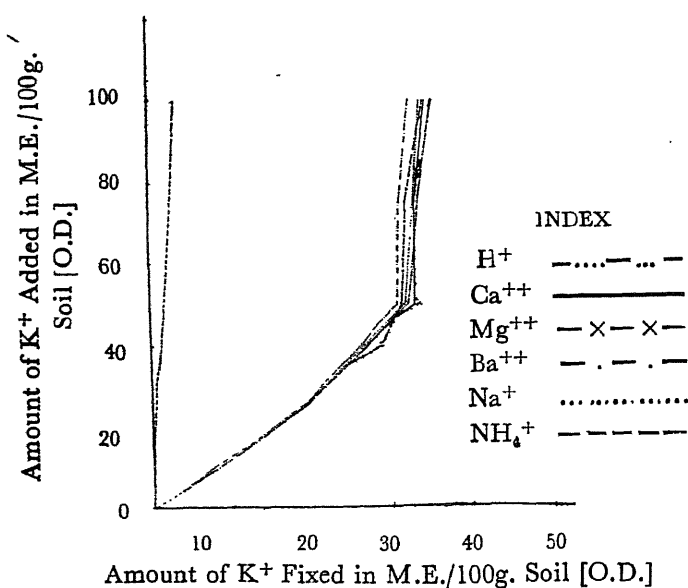


Fig. 2. Amount of K^+ Non-Exchangeable [Fixed] in M. E./100g. Soil [O.D.]

half of the abscissa. It appears from fig. 2 that the amount of K^+ which remained non-exchangeable i. e. fixed increases linearly to a definite point, then it falls off rapidly with the increase in the amount of K^+ added. After that there is rapid fall in the increase of non-exchangeable (fixed) K^+ . Although the amount of fixed K^+ increases with the increase in the amount of K^+ applied, the rate of increase is very slow. On the other hand, the amount of K^+ which remains non-exchangeable (fixed) with NH_4^+ is remarkably less. It can be seen in the Table 9, Fig 2, that when a small amount of K^+ is added, almost all adsorbed K^+ is extracted out by NH_4^+ ions. The amount of non-exchangeable (fixed) K^+ with ammonium ion varied from 0.40 to 3.32 m.e. %.

The amount of adsorbed K^+ (exchangeable) with H^+ , Ca^{++} , Mg^{++} , Ba^{++} , Na^+ , and NH_4^+ ions was plotted against the amount of K^+ applied with the ordinate half the abscissa. It can be seen in Fig. 3 and Table 9 that the increase

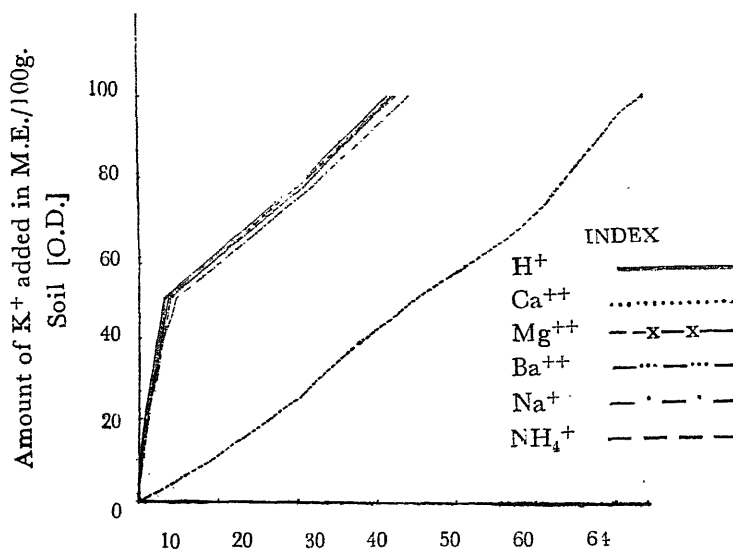


Fig. 3. Amount of K^+ Exchangeable in M.E./100g. of Soil [O.D.]

in the amount of exchangeable K^+ with these ions is very slow upto a point and after this critical point it increases suddenly when further application of K^+ is made. On the other hand, the amount of adsorbed K^+ exchangeable with NH_4^+ increases linearly with the amount of K^+ added.

In order to determine the relationship between the amount of K^+ adsorbed and the amount of K^+ in the exchangeable and non-exchangeable (fixed) forms, graphs were plotted separately. The amount of non-exchangeable (fixed) K^+ by H^+ , Ca^{++} , Mg^{++} , Ba^{++} , Na^+ and NH_4^+ ions was plotted against the amount of K^+ adsorbed with the ordinates and abscissa of equal scales. In this case also it is evident from Fig. 4 that the amount of non-exchangeable (fixed) K^+ increases linearly with the increase in the amount of K^+ adsorbed upto a definite point, then it falls off suddenly. And after this critical point, although the amount of non-exchangeable (fixed) K^+ increases, it does so with a decreasing rate. It will be noted here that there is a close similarity between Fig. 2 and Fig. 4. This shows that it is the amount of adsorbed K^+ which governs the fate of K^+ in the exchangeable and the non-exchangeable form. It appears that the similarity found in Fig. 4 in the behaviour of H^+ , Ca^{++} , Mg^{++} , Ba^{++} and Na^+ ions and dissimilarity in the behavior of NH_4^+ ion in replacing the adsorbed K ions is observed here also. The amount of K^+ in exchangeable form was also plotted against the amount of K^+ adsorbed, with the ordinates and abscissa of equal scales. A close similarity between Fig. 3 and Fig. 5 is observed.

The slow increase in the amount of exchangeable K^+ and the linear increase in the amount of non-exchangeable K^+ indicate that as long as the fixing capacity of the soil is not satisfied most of the adsorbed K ions are being fixed and a very

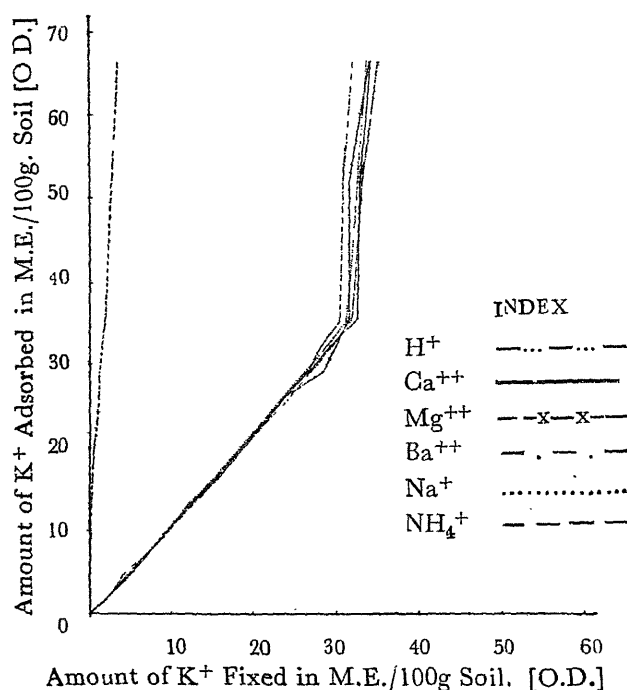


Fig. 4. Amount of K^+ Non-Exchangeable [Fixed] in ME/100g Soil [O.D.]

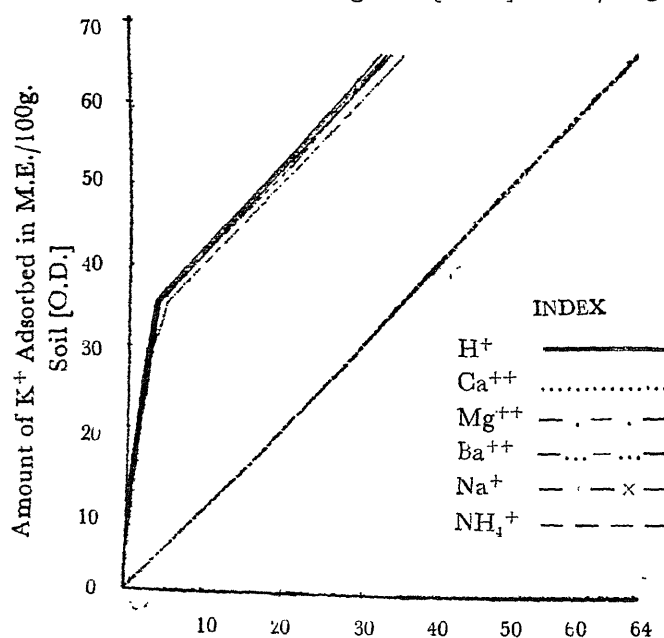


Fig. 5. Amount of 'K' Exchangeable in M.E./100g. Soil [O.D.]

small quantity remains in the exchangeable form. And as soon as the fixing capacity is satisfied the amount of exchangeable K^+ becomes larger.

Barshad (4), Hendricks, Nelson and Alexander (19), Jackson and Hellman (25) observed a hydrated nature of Ca^{++} , Mg^{++} , Ba^{++} and Na^+ ions and the unhydrated nature of K and NH_4 ions at the exchange complex of soil or clay. Due to their hydrated nature, ions like Ca^{++} , Mg^{++} , Ba^{++} and Na^+ may not enter the lattice "hole" with a diameter of 2.8 \AA . Thus these ions are unable to replace the fixed K ions. On the other hand, NH_4 ion having nearly equal ionic diameter as that of K and being unhydrated, can easily penetrate the "lattice hole" (the space between the two sheets of hexagonally placed oxygen atoms), as described by Page and Baver (32) and can easily replace the fixed K ions which are in a non-replaceable form by H^+ , Ca^{++} , Mg^{++} , Na^+ and Ba^{++} ions. The fore-going discussion indicates that the fixed K^+ can easily be made replaceable even under field conditions by NH_4^+ ions and vice versa, provided there is thorough leaching in the soil column and free movement of ions.

The results obtained for the fixation of K^+ in the soil saturated with different ions like H^+ , Ca^{++} , Mg^{++} , Ba^{++} , Na^+ and NH_4^+ prior to fixation are shown in the Table 10. The fixed K^+ recovered in the H^+ saturated soil prior to fixation was very small, only 2.03 m.e./100 g. soil with N KCl as fixing source as compared to 28.21 m.e./100g. soil when alcoholic KOH was used as the K^+ fixing source. But when the soil was pre-saturated with Ca^{++} , Mg^{++} , Ba^{++} and Na^+ the fixed K^+ released was much greater as compared with the H^+ saturated one, when N KCl was the fixing source. Still greater recovery of fixed K^+ was obtained when the alcoholic K^+ was used as the fixing source. This shows that the fixation of K^+ at a high pH exceeds the quantity of K^+ fixed in the neutral or in the acidic condition in soil. It is interesting to note that with the ammonium ion saturated soil the fixation of K^+ was higher with the neutral normal KCl as well as with the alcoholic KOH treated soils. Stanford (39) found considerable increase in the amount of K^+ fixed by increasing the pH with either NaOH or $Ca(OH)_2$ in illite minerals. It has been frequently reported that lime addition stimulates the fixation of K^+ . MacIntire *et al* (31), investigating the effect of long term lime and magnesia treatments upon the outgo of the subsequent addition of K^+ found that the addition of lime to the soil increased the fixation of K^+ . Harris (23) has reported that hydrated lime results in an increase in the quantity of K^+ a soil will fix. The work of Wrenshall and Marcello (48) confirms the observations of Harris (23) Gilligan (18) also noted, that K^+ fixation increases with the increasing of pH. Peech and Bradfield (35), demonstrated that increasing the quantity of Ca^{++} in the exchange complex favours the exchange adsorption of K^+ . Ayres (3) showed that with increasing pH, more K^+ enters the exchange complex and consequently more K^+ is fixed. Joffe and Levine (27) concluded from their investigations that increase in the fixation of K^+ by lime addition is due to the increase in the pH of the system, by means of which more K^+ is able to enter the exchange complex rather than to Ca^{++} itself. Peterson and Jennings (36) also obtained similar results. A conclusion can be drawn from the present investigation that soil having high pH values or having being saturated with bases like Ca^{++} , Mg^{++} , Ba^{++} and Na^+ are capable of fixing K^+ ion while acid soils are less susceptible to fix potassium.

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IODOMETRIC DETERMINATION OF NITROSO GROUP IN SEMIMICRO SCALE

By

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ABSTRACT

An iodometric procedure for the semimicro determination of nitroso group in aromatic nitroso compounds has been presented. Nitroso compounds liberate an equivalent amount of iodine from an acidified solution of potassium iodide. 20-30 mg sample was dissolved in glacial acetic acid or purified ethanol and treated with 10.0 ml (10%) potassium iodide solution in presence of 5.0 ml of 6N hydrochloric acid. After 20 minutes liberated iodine was titrated with standard sodium thiosulphate solution to the starch end point.

Blank determination on exactly equal amounts of reagents was carried out under identical conditions.

Nitroso compounds can be reduced with an acidified solution of potassium iodide, liberating an equivalent amount of iodine which can be titrated with sodium thiosulphate solution. The importance of this reaction lies in the fact that aromatic nitro compounds do not undergo this reaction and hence determination of nitroso groups can be done even in the presence of nitro compounds. Iodometric determination of diphenyl nitrosoamine has already been reported by Becker and Shaefer¹. A similar iodometric procedure for nitrosobenzene has also been reported by Lobunets and Gortins'ka². Efforts have been made in the present work to adopt this procedure in the semimicro and micro scales. Though the method does not seem promising in general for all nitroso compounds, fairly good results can be obtained in the case of some nitroso compounds in the semimicro scale. The procedure did not yield satisfactory results in the micro scale. Since mineral acids also liberate iodine from potassium iodide solution, performance of blank on the exactly equal amounts of reagents run for the equal amounts of time, is essential. Blank values increased with increasing time.

The reaction time also varied with different compounds. 1-Nitroso 2-naphthol, 6-nitroso *m*-cresol and 5-nitroso *o*-cresol reacted quantitatively in 15-20 minutes while 2-nitroso 1-naphthol gave values which were about 2% low even after 45-60 minutes of reaction period. 4-Nitroso 1-naphthol did not give satisfactory results even after reaction had been allowed to proceed for more than one and half hours.

Colour of the reaction products interfered with the starch end point in some cases. 2-Nitroso 1-naphthol, 6-nitroso *m*-cresol and 5-nitroso *o*-cresol gave clear end points while 1-nitroso 2-naphthol and 4-nitroso 1-naphthol gave brownish end point. However, after a little practice it would not be difficult to locate the correct end point.

All the determinations were carried out in aqueous solutions in presence of nitrogen gas. 3-4 ml of glacial acetic acid was used as solvent. Purified ethanol may be used for the substances which have low solubility in acetic acid. The

results of determinations in the semimicro scale on five nitroso compounds are given in table 1.

EXPERIMENTAL

Reagents:—

6N hydrochloric acid (A. R.), 10% solution of potassium iodide (M. A. R., H. and W.), 0.05N solution of sodium thiosulphate (A. R., B. D. H.), Glacial acetic acid (A.R., B.D.H.), purified ethanol, 1% solution of starch (A. R., B. D. H.) prepared afresh each day.

TABLE I

Percentage of nitroso group determined iodometrically in the semimicro scale.

Compound	Percentage of nitroso group		
	Cal.	Found	Mean % error
1-Nitroso 2-Naphthol	17.34	16.93, 16.76	-0.50
2-Nitroso 1-naphthol	17.34	15.34, 15.28	-2.03
4-Nitroso-1-naphthol	17.34	14.00, 13.36	-3.66
6-Nitroso <i>m</i> -cresol	21.87	21.46, 21.24	-0.52
5-Nitroso <i>o</i> -cresol	21.87	20.92, 20.70	-1.06

Procedure:

20-30 mg sample was accurately weighed and carefully transferred to the reaction flask provided with an arrangement for passing nitrogen gas. The sample was dissolved in 3-4 ml of glacial acetic acid (or purified ethanol) and 5.0 ml of 6N hydrochloric acid added. The air inside the flask was removed by passing nitrogen gas for few minutes; 10.0 ml of potassium iodide solution added, nitrogen supply cut off and the flask stoppered and set aside in dark for 20 minutes. After the reaction was over, liberated iodine was titrated with 0.05N sodium thiosulphate solution to the starch end point.

A concurrent blank on the exactly equal amount of reagents, except the sample, was run under identical conditions and the blank value subtracted from the reading.

ACKNOWLEDGEMENT

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REACTION BETWEEN *p*-TOLUIDINE AND *m*-DINITROBENZENE

By

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ABSTRACT

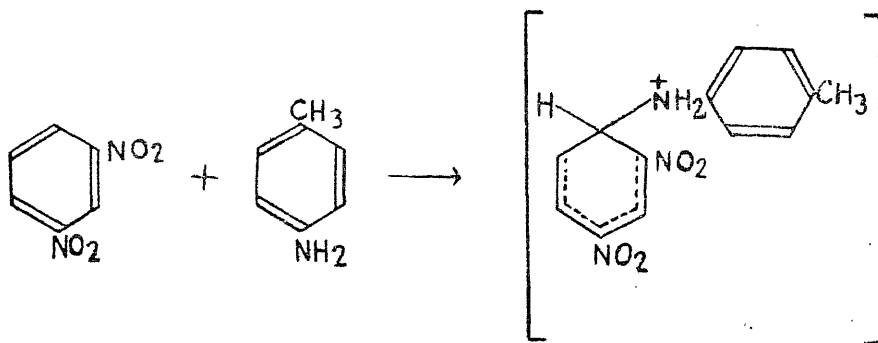
The reaction between *p*-toluidine and *m*-dinitrobenzene has been studied and the colour formation when the two substances are mixed in the solid state or in the ethanol solutions has been explained on the basis of the intermolecular charge-transfer complex formation and hyperconjugation.

When solid samples of *p*-toluidine and *m*-dinitrobenzene are mixed together, an intense orange colour develops immediately and the two components melt. The addition of a suitable solvent such as ethanol to the mixture does not destroy the colour, though on account of dilution the colour appears yellowish in the solution. If the separate solutions of *p*-toluidine and *m*-dinitrobenzene in ethanol are mixed together, the yellow colour develops immediately in this case as well. The behaviour of some aromatic nitro compounds towards aromatic and aliphatic amines has been investigated by Mulliken¹ and others^{2,3,4} and it has been shown that in all such cases intermolecular charge-transfer complexes are obtained. While in the case of interaction between aromatic nitro compounds and aromatic amines the complexes show a single featureless absorption band in near ultraviolet or visible region, the interaction of aromatic nitro compounds and aliphatic amines produces complexes which may give more than one absorption band changing in shape and intensity with time^{3,4,5}. It appears to be necessary that for the charge-transfer absorption the donor should have a high-energy filled orbital and the acceptor a low-energy vacant orbital and these orbitals should overlap.⁶

The case of interaction between *p*-toluidine and *m*-dinitrobenzene appears to be influenced by another factor also and that is hyperconjugation or no-bond resonance in *p*-toluidine. Absorption measurement of the complex formed in ethanol shows that the region of absorption is mostly in between 320 to 400 $m\mu$. It is reasonable to assume that in the region (300—400 $m\mu$) absorption characteristics of the unreacted components are also present particularly on the lower wavelength region (near 320 $m\mu$).

The position which is *o*- or *p*- with respect to both the nitro groups is expected to be highly reactive for this reaction due to the electron attracting nature of the two nitro groups. Thus considering the above facts and the similar observations for the other groups of donors and acceptors by other workers mentioned above,

it can be tentatively suggested that the stoichiometry between the components in solutions in this reaction is 1 : 1 and the structure of the complex is :



I

The stoichiometry of 1 : 1 may also be expected due to the steric factors. The detailed study of the reaction in respect of the nature of the complex, stability and also with other nitro compounds is in progress.

EXPERIMENTAL

Materials—*m*-Dinitrobenzene (M. A. R., Hopkins and Williams Ltd. England), *p*-Toluidine (AnalaR, B. D. H. England), ethanol absolute pure.

Apparatus—Beckman model DU Spectrophotometer equipped and attached with a multiplier phototube operated at varying sensitivity. Cells - Beckman Corex 1.005 cm cells.

Solutions of *m*-dinitrobenzene and *p*-toluidine were prepared separately and the components mixed. All the absorption measurements were done at room temperature.

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STABILITY OF ARSENIOUS SULPHIDE SOL OF DIFFERENT PARTICLE SIZES : OPTICAL STUDY

By

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ABSTRACT

Three samples of arsenious sulphide sols of different particle sizes were obtained from a polydispersed sol by centrifuging through a laboratory Sharple's Supercentrifuge at 6,000, 12,000 and 20,000 r. p. m. The stability of the sols was investigated employing a Klett Summerson photoelectric colorimeter. It has been shown that the sol sample having particles of biggest average size obtained by centrifuging at the lowest speed viz. 6,000 r. p. m. is least stable. The stability gradually increases with the decrease of the average particle size. It has been further shown that the presence of dioxan decreases the stability, while alcohol brings about an increase. Bhattacharya's equation is applicable for the process of coagulation.

Several optical methods have been used by various workers to study the rate of coagulation of different sols. Mukherjee and coworkers¹ used optical methods for studying the rate of coagulation of gold sols, while Ghosh and Rai² made similar investigations with silver sols. Prasad and Modak³ have studied the velocity of coagulation of sols by following opacity changes during coagulation. Tezak⁴ made tyndallometric observations to follow the process of coagulation of silver halide sols. Chatterji and Tewari⁵ used Spekker photoelectric absorptiometer in their investigations on the effect of non-electrolytes on several sols. Prakash⁶ investigated the extinction coefficients of several inorganic gels, and Mushran and Prakash⁷ investigated the negatively charged colloidal solutions of various ferric salts employing Nuttings photoelectric colorimeter. Ottewill and Watanabe⁸ have investigated the stability of positive silver iodide sols in presence of anionic surface active substances employing unicam SP 600 spectrophotometer.

Optical methods of studying the coagulation of sols have the advantage in the fact that they provide us with a fair idea of the rate of coagulation of sols by different concentrations of the coagulating electrolyte and also of the different stages of aggregation reached in the coagulation process at different intervals of time. Bhattacharya⁹ has tested the validity of Bhattacharya's equation making use of the time taken by different concentrations of the coagulating electrolyte to reach the same stage of aggregation in the process of the precipitation of silver halide sol.

We have here studied the coagulation and sensitisation of negatively charged arsenious sulphide sol employing Klett-Summerson photoelectric colorimeter. Effect of particle size and effect of non-electrolytes on the stability of the sols have been investigated. Attempt has also been made to test the validity of Bhattacharya's equation. Of the large number of observations made only a few have been tabled or plotted here to illustrate and discuss the results from the point of view of the above effects.

EXPERIMENTAL

A fresh sample of negatively charged arsenious sulphide sol was prepared by the usual method. It was centrifuged employing Sharples' Laboratory Supercen-

trifuge at three different speeds viz., 6,000; 12,000 and 20,000 r. p. m. to obtain three samples A, B and C respectively, differing in the average sizes of the colloid particles. The sulphide content in each sample was made the same (1.71 gm. As_2S_3 / litre) after due estimation and proper dilution.

For purpose of study 2.0 ml of sol was taken each time in Klett test tubes. Different amounts of the coagulating electrolyte or the coagulating electrolyte plus non-electrolyte mixture were taken in another series of test tubes and the total volume was kept 6.0 ml, in each case. The electrolyte used as coagulant was potassium chloride and the non-electrolytes used were dioxan and ethyl alcohol, all of A. R. quality. Filter No. 42 supplied with the instrument was employed for optical density measurements.

OBSERVATIONS AND RESULTS

Colorimetric readings obtained with sol A coagulated by different amounts of potassium chloride have been shown in Table I. The readings obtained with sols A, B and C, coagulated by a definite amount of potassium chloride have been placed in Table 2. Table 3 records the same values for sol A coagulated by the same amount of electrolyte in presence of dioxan and ethyl alcohol, each of 2% concentration in the total volume. In Tables 4 and 5 the colorimetric readings for all the three samples of the sol coagulated by the same amount of potassium chloride in presence of the same concentration of the non-electrolytes have been respectively compiled.

The process of coagulation of sol A with different concentrations of the coagulating electrolyte has been graphically shown in Fig. 1. Since the colorimetric reading multiplied by the factor 0.002 gives the optical density ($\log I_0/I_t$) at any time of observation, such readings have been directly plotted against time.

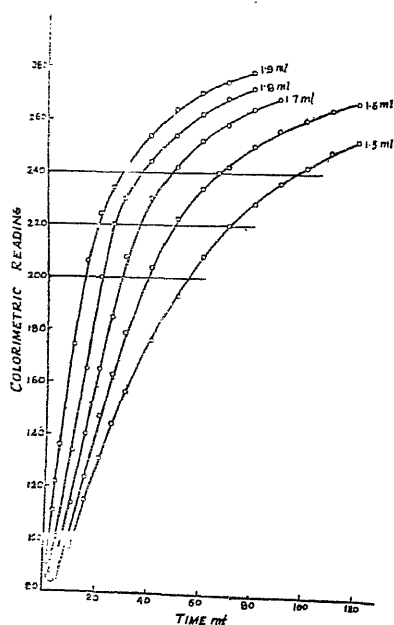


Fig. 1. Plot of Colorimeter reading vs. time : Sol A + KCl ,

For purpose of testing the applicability of Bhattacharya's equation

$$\frac{1}{C - a} = \frac{n}{m} \cdot t + \frac{1}{m},$$

the following procedures were adopted. The times corresponding to the same stage of coagulation with different concentrations of the electrolyte were determined by drawing a line parallel to the time axis in the colorimetric reading (CR) *vs.* time curve (Fig. 1) and these have been placed in Tables 6 and 7. Having thus known the times (*t*) of coagulation by different concentrations (*c*) of the electrolyte for the same stage of coagulation, the *C vs. 1/t* curve was plotted (Fig. 2).

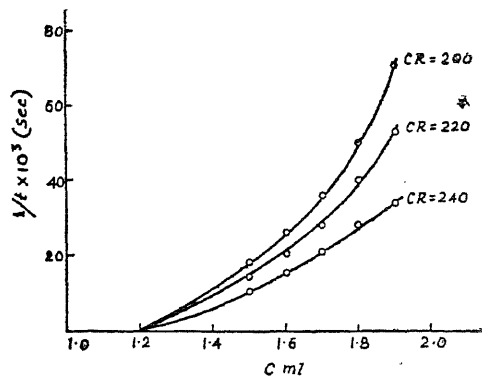


Fig. 2. Plot of *c vs. 1/t* : Sol A + KCl.

Three such curves have been shown for three different stages of coagulation. After knowing the value of the constant *a* from this curve, $1/(C - a)$ was now plotted against *t* (Fig. 3) when straight lines were obtained for all the three cases. The values of the constants *m* and *n* were evaluated as usual and recorded in Table 8.

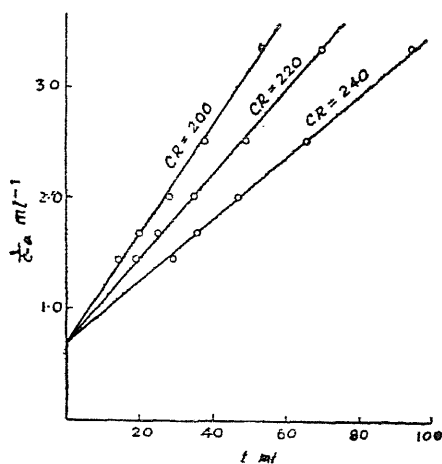


Fig. 3. Plot of $1/(c - a)$ *vs. t* : Sol A + KCl.

TABLE 1

Coagulation of Sol A by 0.5 M Potassium Chloride

Time (min.)	Colorimetric Reading				
	Volume of KCl (ml.)				
	1.9	1.8	1.7	1.6	1.5
1	91	80	73
2	100	85	80
3	111	91	84	75	...
4	122	95	87	80	...
5	136	101	92	85	76
10	174	134	114	101	97
15	206	165	140	124	115
20	224	200	165	147	131
25	234	220	185	163	144
30	242	230	208	179	156
40	254	244	230	204	176
50	264	254	242	222	193
60	270	262	252	234	208
70	274	268	258	242	220
80	278	272	264	250	228
90	268	256	236
100	260	242
110	264	248
120	266	252

TABLE 2
Coagulation of Sols A, B and C by 1.9 ml of 0.5 M Potassium Chloride

Time (min)	Colorimetric Reading		
	Sol A	Sol B	Sol C
1	91	91	89
2	100	96	93
3	111	104	99
4	122	112	106
5	136	121	113
10	174	164	150
15	206	193	179
20	224	214	208
25	234	228	222
30	242	238	232
40	254	250	240
50	264	260	256
60	270	268	264
70	274	272	270
80	278	276	274

TABLE 3
Coagulation of Sol A by 1.9 ml of 0.5 M Potassium Chloride in Presence of
Non-electrolytes

Time (min)	Colorimetric Reading		
	Sol A + 2% Dioxan	Sol A + 0% non-electrolyte	Sol A + 2 ethyl alcohol
1	105	91	...
2	140	100	...
3	169	111	...
4	191	122	78
5	204	136	85
10	236	174	98
15	254	206	120
20	266	224	143
25	274	234	158
30	278	242	168
40	284	254	196
50	290	264	212
60	292	270	228
70	...	274	236
80	...	278	242
90	250

TABLE 4
Coagulation of Sols A, B and C by 1.9 ml of 0.5 M Potassium Chloride in
presence of 2% Dioxan

Time (min)	Colorimetric Reading		
	Sol A	Sol B	Sol C
1	105	107	108
2	140	145	148
3	169	174	180
4	191	199	206
5	204	216	226
10	236	246	258
15	254	266	276
20	266	278	286
25	274	282	294
30	278	284	296
40	284	288	305
50	290	294	310
60	292	296	310

TABLE 5
Coagulation of sols A, B and C by 1.9 ml of 0.5 M Potassium Chloride in
presence of 2% Ethyl alcohol

Time (min)	Colorimetric Reading		
	Sol A	Sol B	Sol C
4	78	75	...
5	85	80	73
10	98	92	85
15	120	113	104
20	143	135	125
25	158	148	137
30	168	158	146
40	196	183	168
50	212	196	178
60	228	212	192
70	236	216	196
80	242	222	202
90	250	230	215

TABLE 6

Values of C and for Sol A at Three Stages of Colorimetric Reading (CR)

0.5 M KCl ml	$1/t \times 10^3$ (Sec.)		
	CR = 200	CR = 220	CR = 240
1.5	18.52	14.29	10.50
1.6	26.32	20.41	15.15
1.7	35.71	28.57	21.28
1.8	50.00	40.00	27.78
1.9	71.43	52.63	34.48

TABLE 7

Values of $\frac{1}{C-a}$ and t for Sol A

$(C-a)$	$\frac{1}{C-a}$	t (min)		
		CR = 200	CR = 220	CR = 240
0.3	3.33	54	70	95
0.4	2.50	38	49	66
0.5	2.00	28	35	47
0.6	1.67	20	25	36
0.7	1.43	14	19	29

TABLE 8

Values of the Constants a , m and n for Sol A

Stage	a	m	$n \times 10^2$
CR = 200	1.2	1.46	7.19
CR = 220	1.2	1.46	5.54
CR = 240	1.2	1.46	4.08

DISCUSSION AND CONCLUSION

A perusal of the data presented in the above tables show that the coagulation of Sol A proceeds at a faster rate than Sol B or Sol C. This means that the sol sample having particles of biggest average size obtained by centrifuging at the lowest speed (6,000 r. p. m.) is least stable and the one having particles of smallest average size obtained by centrifuging at the highest speed (20,000 r. p. m.) is most stable. That is the stability follows the following order

$$\text{Sol C} > \text{Sol B} > \text{Sol A}$$

It is seen that the presence of dioxan in the sol renders it less stable, while ethyl alcohol brings about increased stability. The effect of sensitisation or stabilisation increases with the fineness of the dispersed particles (vide tables 5 and 6). It is further seen from the colorimetric data presented in this paper that the Bhattacharya's equation is applicable for the process of coagulation.

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COMPLEX FORMATION BETWEEN CERIUM (III) AND DIBROMOSUCCINIC ACID

By

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ABSTRACT

Complex formation between dibromosuccinic acid (sodium salt) and cerium (III) has been studied by the monovariation and Job's continuous variation methods, the properties investigated being conductance, pH and optical density. A 1:1 complex is formed in the system. It has been assigned the structural formula, $[\text{Ce}(\text{COOCHBr})_2]^{+}$. Its dissociation constant has been found to be 7.23×10^{-4} . The complex exhibits strong continuous absorption in the ultraviolet region. The range of maximum stability of the complex has been found to be between pH 5 to 6.

Though succinic acid has been reported to form chelates with a number of metals, dibromosuccinic acid does not appear to have been used as a chelating agent. The present work reports our results of a detailed physicochemical study on the system, cerous chloride-sodium dibromo-succinate.

EXPERIMENTAL

A Doran conductivity bridge was used for conductometric measurements, whereas pH measurements were carried out on a Beckman pH meter (model H/2). Absorption studies were made on a Beckman DU-spectrophotometer using a hydrogen lamp. The experiments were carried out at a temperature of 28°C.

A recrystallised sample (B. D. H.) of cerous chloride was used. The cerium content was estimated by precipitation as cerous oxalate, subsequent ignition to ceric oxide and weighing as CeO_2 . Sodium salt of dibromosuccinic acid was prepared by mixing the equivalent amounts of acid (B. D. H.) and anhydrous sodium carbonate (Analar). Concentrated solutions of hydrochloric acid (Analar) and sodium hydroxide were used for adjusting the pH of the solutions. All solutions were prepared in conductivity water.

COMPOSITION OF THE COMPLEX

pH Study:—A series of solutions was prepared by taking a constant volume (5 ml) of M/30 cerous chloride solution, adding to it increasing volumes (from 1 ml to 20 ml.) of equimolar sodium dibromosuccinate solution and making up the total volume of each sample to 25 ml. by the addition of conductivity water. The solutions were kept for some time for attaining equilibrium. The pH values when plotted (Fig. 1 curve A) against the molar ratio, sodium dibromosuccinate/cerous chloride, exhibited a definite break at one, suggesting, thereby, the formation of a 1:1 complex in the system.

Conductometric study:—Measurements, of conductivity and a plot of the observed data yielded curve B (Fig. 1) with a break at 1:1 ligand to metal ratio. Job's method of continuous variation¹ was also applied to confirm the above conclusion. Three sets of equimolar solutions (viz. M/30, M/40 and M/60) of sodium dibromosuccinate and cerous chloride were used for this purpose. The deviations from additivity rule, in each case were plotted against the percentage

of the ligand (Fig. 2). The curves exhibited definite maxima at 50% of the ligand, confirming, thereby, the formation of 1:1 cerous dibromosuccinate complex.

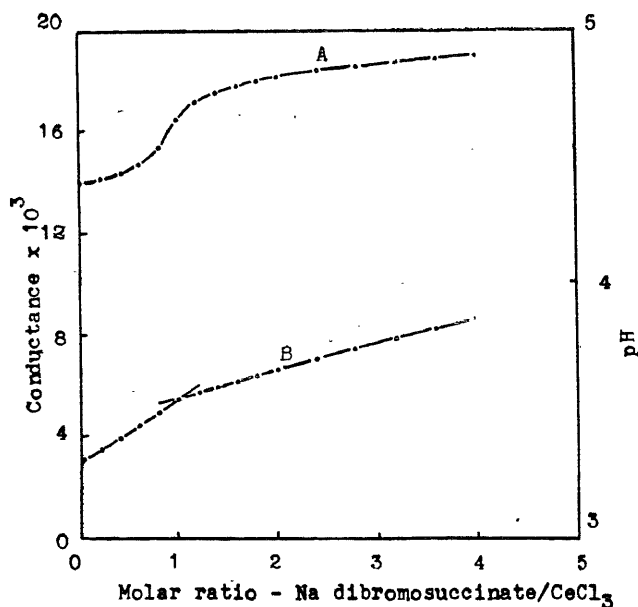


Fig. 1. Composition of the complex by monovariation method using M/30 equimolar solutions. Curve A: pH; Curve B: conductance.

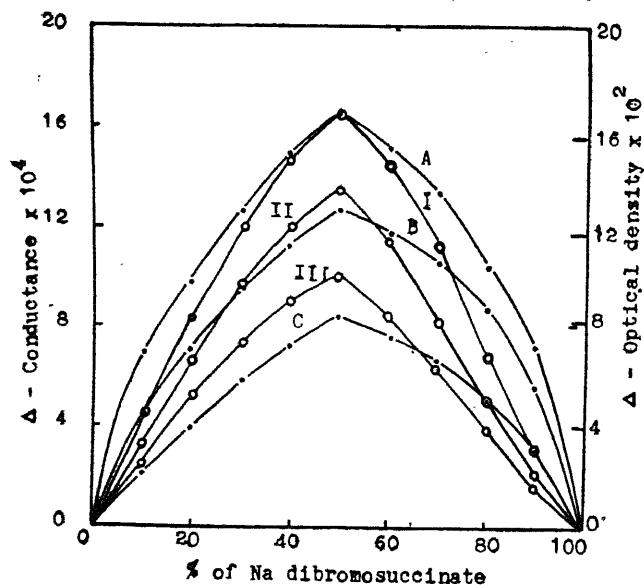


Fig. 2. Composition of the complex by Job's method:
(a) Conductometric study: curves A, B and C.
(b) Spectrophotometric study: curves I, II and III.

Spectrophotometric Study:—Before actually determining the composition of the complex it was thought necessary to study the change in optical density with pH. Various samples of the complex were prepared by mixing cerous chloride and sodium dibromosuccinate solutions in 1:1 ratio. Their pH value was adjusted within a wide range by the addition of a concentrated sodium hydroxide or hydrochloric acid solution. The dilution due to the addition of alkali or acid solution was treated as negligible. The observed optical density was plotted against the pH. The experimental curve (Fig. 3) indicated that the complex was

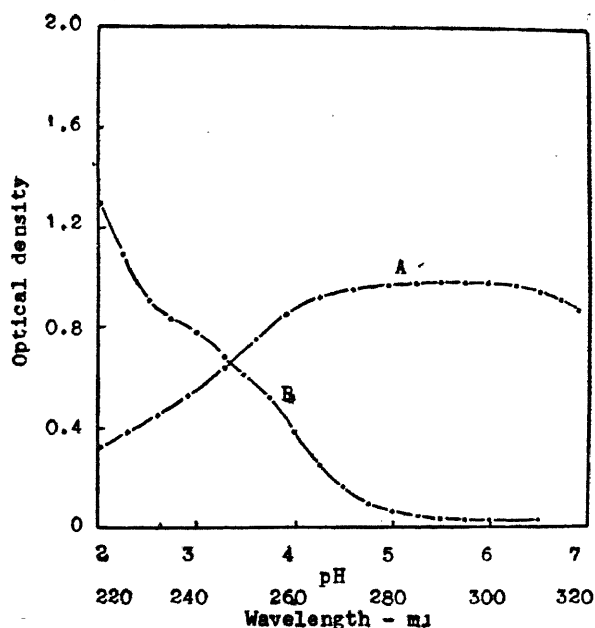


Fig. 3. Curve A: Variation of O. D. with pH at 250 $m\mu$ using M/500 solutions mixed in 1:1 ratio.
Curve B: Variation of O. D. with wavelength using M/600 solutions mixed in 1:1 ratio.

stable at a pH, range of 5-6. In the spectrophotometric measurements, therefore, the pH of the various solutions was kept constant within this range.

The variation of optical density with wavelength was also studied. The system exhibited continuous absorption (Fig. 3) in the ultraviolet range. However, the measurements were carried out at 250 $m\mu$, 260 $m\mu$ and 270 $m\mu$. Application of Job's method using equimolar solutions of the reactants yielded three curves each with a maximum at 50% of sodium dibromosuccinate (Fig. 2). This was in perfect accord with the earlier observations.

Dissociation constant of the complex:—The dissociation constant of the complex was determined by an application of Job's method to nonequimolar solutions of the reactants. Conductivity measurements were carried out with three sets of nonequimolar solutions (viz M/60 cerium+M/30 ligand, M/60 cerium+M/40 ligand and M/80 cerium+M/40 ligand) and the spectrophotometric measurements with M/600 cerium and M/400 ligand solutions at 250 $m\mu$, 260 $m\mu$ and 270 $m\mu$. Both the conducto-

metric as well as spectrophotometric curves (Fig. 4) gave consistent values of the dissociation constant, its average value being 7.23×10^{-4} .

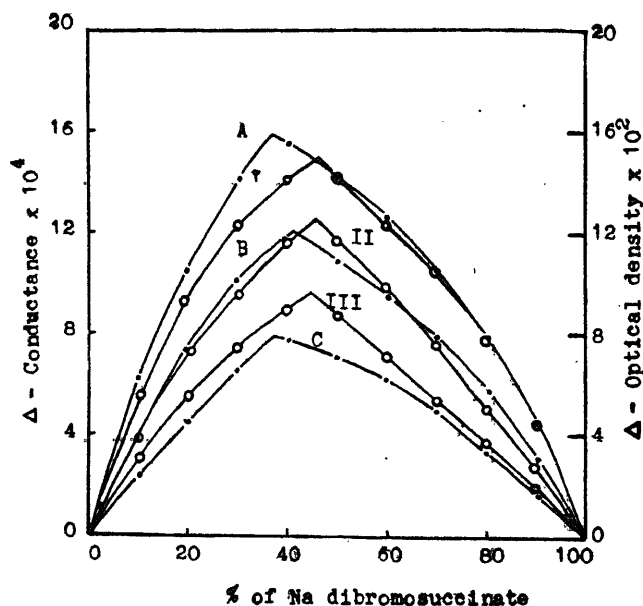
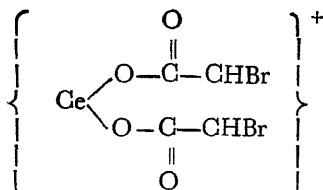


Fig. 4. Dissociation constant of the complex by Job's method.

(a) Conductometric study: Curves A, B and C.

(b) Spectrophotometric study: Curves I, II and III.

The following structure has been proposed for the cerous dibromosuccinate chelate:



The rise in pH (as seen in Fig. 1 curve A) may be attributed to the liberation of sodium ions from sodium dibromosuccinate. The rest of the six coordination positions of cerium (III) in the above chelate are considered to be occupied by water molecules.

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ON KERNELS IN GENERALISED HANKEL TRANSFORM

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ABSTRACT

Mehra (5) has introduced a generalisation of the Hankel transform in the form

$$g(x) = \left(\frac{1}{2}\right)^v \int_0^\infty (xy)^{v+\frac{1}{2}} \chi_{v,k+\frac{1}{2},m} \left(\frac{1}{2} x^2 y^2\right) f(y) dy,$$

I have obtained some kernels, which are the resultant of two kernels and one is the resultant of two functions in this paper. Following in the main, the method of Agarwal, various particular cases and certain suitable examples have been included.

1. A generalisation of the well-known Hankel transform

$$g(x) = \int_0^\infty \sqrt{xy} J_\nu(xy) f(y) dy, \quad (1.1)$$

has been given by Mehra⁵ in the form:—

$$g(x) = 1/2^v \int_0^\infty (xy)^{v+\frac{1}{2}} \chi_{v,k+\frac{1}{2},m} \left(\frac{1}{2} x^2 y^2\right) f(y) dy, \quad (1.2)$$

where $\chi_{v,k+\frac{1}{2},m}(x) = \sum_{m,-m} \frac{\Gamma(2m) \Gamma(v+1-m-3k)}{\Gamma(m-k) \Gamma_x(v+1-m-2k \pm m)} x^{-k-m} \times$

$${}_2F_3 \left[\begin{matrix} 1+k-m, v+1-m-3k \\ 1-2m, v+1-m-2k+m \end{matrix} ; -x \right] \quad (1.3)$$

provided that $R(v-2k)+1 > |R(2m)|$, $R(s) \geq 0$, $S_0 > 0$ and $2m$ is not an integer or zero.

He calls $g(x)$ in (1.2) to be the $\chi_{v,k+\frac{1}{2},m}$ -transform $f(x)$ and has shown that this transform is reciprocal like the Hankel transform. Thus, if $g(x)$ is the $\chi_{v,k+\frac{1}{2},m}$ -transform of $f(x)$, so is $f(x)$ of $g(x)$. In case $g(x) = f(x)$ i. e. if $f(x)$ is its own $\chi_{v,k+\frac{1}{2},m}$ -transform, we may say that $f(x)$ is self-reciprocal for the $\chi_{v,k+\frac{1}{2},m}$ -transform, and may be denoted by $R_v(k+\frac{1}{2},m)$. It may be noted that on putting $k=-m$, (1.1) reduces to (1.1).

Mehra has given certain kernels in this transform connecting different classes of self-reciprocal functions. Some more kernels have been given by the present author⁸. The object of this paper is to investigate some more kernels and also to find out some kernels which are resultants of two functions in this transform and others which are resultants of two kernels.

2. The necessary and sufficient condition for a function $f(x)$ of $A(\alpha, a)$ to be $R_{\nu}(k+\frac{1}{2}, m)$ is that it should be of the form:

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{s/2} \frac{\Gamma x(\nu/2+s/2+1/4-k\pm m)}{\Gamma(\nu/2+s/2+1/4-2k)} \psi(s) x^{-s} ds, \quad (2.1)$$

where $\psi(s)$ is regular and satisfies the relation

$$\psi(s) = \chi(1-s) \quad (2.2)$$

in the strip $a < 1-\sigma < a$. (2.3)

3. Theorem 1. If $f(x)$ is $R_{\nu}(k+\frac{1}{2}, m)$, and

$$P(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma x(\mu/2-s/2+3/4-l\pm n)}{\Gamma(\nu/2+s/2+1/4-2k)} \frac{\Gamma x(\nu/2+s/2+1/4-k\pm m)}{\Gamma(\mu/2-s/2+3/4-2l)} \omega(s) e^{sx} ds, \quad (3.1)$$

$$= \begin{matrix} x > 0, \\ x < 0, \end{matrix}$$

where c is any positive number and $\omega(s)$ satisfies (2.2) then

$$g(x) = \frac{1}{x} \int_0^x P\left(\log \frac{x}{y}\right) f(y) dy, \quad (3.2)$$

is $R_{\mu}(l+\frac{1}{2}, n)$.

Proof: Since $f(x)$ is $R_{\nu}(k+\frac{1}{2}, m)$, we have

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{s/2} \frac{\Gamma x(\nu/2+s/2+1/4-k\pm m)}{\Gamma(\nu/2+s/2+1/4-2k)} \chi(s) x^{-s} ds,$$

$$\text{and } g(x) = \frac{1}{x} \int_0^x P\left(\log \frac{x}{y}\right) dy \cdot \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{s/2} \frac{\Gamma x(\nu/2+s/2+1/4-k\pm m)}{\Gamma(\nu/2+s/2+1/4-2k)} \psi(s) y^{-s} ds,$$

Changing the order of integration, which is permissible, we get

$$g(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s/2} \Gamma(x(v/2+s/2+1/4-k\pm m))}{\Gamma(v/2+s/2+1/4-2k)} \chi(s) ds \frac{1}{x} \int_0^x P\left(\log \frac{x}{y}\right) y^{-s} dy.$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s/2} \Gamma(x(v/2+s/2+1/4-k\pm m))}{\Gamma(v/2+s/2+1/4-2k)} \psi(s) x^{-s} ds \int_0^\infty e^{(s-1)u} P(u) du.$$

Now, using a form of Mellin's Inversion formula, we get from (3.1)

$$\int_0^\infty e^{-su} P(u) du = \frac{\Gamma(x(\mu/2+3/4-s/2-l\pm n)) \Gamma(x(v/2+s/2+1/4-k\pm m))}{\Gamma(\mu/2-s/2+3/4-2l) \Gamma(v/2+s/2+1/4-2k)} \omega(s),$$

On writing $1-s$ for s , we obtain

$$\int_0^\infty e^{(s-1)u} P(u) du = \frac{\Gamma(x(v/2+3/4-s/2-k\pm m)) \Gamma(x(\mu/2+s/2+1/4-l\pm n))}{\Gamma(v/2-s/2+3/4-2k) \Gamma(\mu/2+s/2+1/4-2l)} \omega(s);$$

as $\omega(s)$ satisfies (2.2)

Therefore,

$$g(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s/2} \Gamma(x(v/2+s/2+1/4-k\pm m)) \Gamma(x(\mu/2+s/2+1/4-l\pm n))}{\Gamma(v/2+s/2+1/4-2k) \Gamma(v/2-s/2+3/4-2k)} \times$$

$$\times \frac{\Gamma(x(v/2+3/4-s/2-k\pm m))}{\Gamma(\mu/2+s/2+1/4-2l)} \chi(s) \omega(s) x^{-s} ds.$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s/2} \Gamma(x(\mu/2+s/2+1/4-l\pm n))}{\Gamma(\mu/2+s/2+1/4-2l)} \phi(s) x^{-s} ds,$$

$$\text{where } \phi(s) = \frac{\Gamma(x(v/2+s/2+1/4-k\pm m)) \Gamma(x(v/2+3/4-s/2-k\pm m))}{\Gamma(v/2+s/2+1/4-2k) \Gamma(v/2-s/2+3/4-2k)} = \phi(1-s).$$

Hence $g(x)$ is $R_{\mu(l+\frac{1}{2}, n)}$

There are two theorems converse to this Theorem.

(a) If $f(x)$ is $R_{v(k+\frac{1}{2}, m)}$ and $g(x)$ is $R_{\mu(l+\frac{1}{2}, n)}$, then $P(x)$ will be given by (3.1).

Proof is similar to that of the above Theorem.

(b) If $g(x)$ is $R_{\mu}(l+\frac{1}{2}, n)$ and $P(x)$ is given by (3.1), then $f(x)$ is $R_{\nu}(k+\frac{1}{2}, m)$.

The proof follows from the theorem itself.

4. *Corollary to Theorem L:*—If in the theorem $\mu=\nu$, we have a theorem like this:

If $f(x)$ is $R_{\nu}(k+\frac{1}{2}, m)$, the function

$$g(x) = \frac{1}{x} \int_0^x P\left(\log \frac{x}{y}\right) f(y) dy, \quad (4.1)$$

is $R_{\nu}(k+\frac{1}{2}, m)$, provided that

$$\begin{aligned} P(x) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{sx} \omega(s) ds, & x > 0 \\ &= 0 & x < 0; \end{aligned} \quad (4.2)$$

where $\omega(s)$ satisfies (2.2).

There are two converse theorems to this corollary also.

(a) If $f(x)$ is $R_{\nu}(k+\frac{1}{2}, m)$ and $g(x)$ given by (4.1) is $R_{\nu}(k+\frac{1}{2}, m)$, then $P(x)$ will be given by (4.2).

(b) If $P(x)$ is given by (4.2) and $g(x)$ is $R_{\nu}(k+\frac{1}{2}, m)$ then $f(x)$ is also $R_{\nu}(k+\frac{1}{2}, m)$.

5. *Theorem 2:*—If $f(x)$ is $R_{\nu}(k+\frac{1}{2}, m)$, and (5.1)

$$\text{then } g(x) = \int_0^1 P\left(\log \frac{1}{xy}\right) f(y) dy \quad (5.2)$$

is $R_{\mu}(l+\frac{1}{2}, n)$ and $\omega(s)$ satisfies (2.2).

Proof of this theorem follows from theorem 1.

There will be two theorems converse to this theorem too.

(a) If $f(x)$ is $R_{\mu}(k+\frac{1}{2}, m)$ and $g(x)$ given by (5.2) is $R_{\mu}(l+\frac{1}{2}, n)$, then $P(x)$ will be given by (5.1).

(b) If $P(x)$ is given by (5.1) and $g(x)$ given by (5.2) is $R_{\mu}(l+\frac{1}{2}, n)$, then $f(x)$ will be $R_{\nu}(k+\frac{1}{2}, m)$.

6. Mehra has given the theorem:

If $f(x)$ is $R_{\nu}(k+\frac{1}{2}, m)$ and

$$P(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^s \Gamma(x/2+s/2+1/4-k\pm m) \Gamma(x/2+s/2+1/4-l\pm n)}{\Gamma(\nu/2+s/2+1/4-2k) \Gamma(\mu/2+s/2+1/4-2l)} \omega(s) x^{-s} ds, \quad (6.2)$$

$$\text{where } \omega(s) = \omega(1-s), \quad (6.1)$$

$$\text{then } g(x) = \int_0^{\infty} p(xy) f(y) dy, \quad (6.2)$$

is $R_{\mu}(l+\frac{1}{2}, n)$.

There are two theorems converse to this theorem also.

(a) If $f(x)$ is $R_{\nu}(k+\frac{1}{2}, m)$, and $g(x)$ given by (6.2) is $R_{\mu}(l+\frac{1}{2}, n)$, then $P(x)$ will be given by (6.1).

Proof:—Multiplying (6.2) by x^{s-1} and integrating between $(0, \infty)$, we get

$$\int_0^{\infty} g(x) x^{s-1} dx = \int_0^{\infty} f(y) P(xy) dy \int_0^{\infty} x^{s-1} dx.$$

Changing the order of integration, which is possible,

we get

$$\int_0^{\infty} x^{s-1} g(x) dx = \int_0^{\infty} f(y) dy \int_0^{\infty} x^{s-1} P(xy) dx.$$

On writing u for xy , we have

$$\int_0^{\infty} x^{s-1} g(x) dx = \int_0^{\infty} y^{-s} f(y) dy \int_0^{\infty} u^{s-1} P(u) du, \quad (6.3)$$

Now, since $f(x)$ is $R_{\nu}(k+\frac{1}{2}, m)$ and $g(x)$ is $R_{\mu}(l+\frac{1}{2}, n)$, we have

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s/2} \Gamma(x/2+s/2+1/4-k\pm m)}{\Gamma(\nu/2+s/2+1/4-2k)} \psi(s) x^{-s} ds,$$

where $\psi(x)$ is regular and satisfies (2.2).

Applying Mellin's Inversion formula to this, we get

$$\int_0^{\infty} x^{s-1} f(x) dx = \frac{2^{s/2} \Gamma x (v/2+s/2+1/4-k \pm m)}{\Gamma (v/2+s/2+1/4-2k)} \psi (s).$$

Writing $1-s$ in place of s , we obtain

$$\int_0^{\infty} x^{-s} f(x) dx = \frac{2^{1/2-s/2} \Gamma x (v/2-s/2+3/4-k \pm m)}{\Gamma (v/2-s/2+3/4-2k)} \psi (s).$$

Similarly

$$\int_0^{\infty} x^{s-1} g(x) dx = \frac{2^{s/2} \Gamma x (\mu/2-s/2+1/4-l \pm n)}{\Gamma (\mu/2+s/2+1/4-2l)} \phi (s).$$

Therefore, from (6.3) we have

$$\begin{aligned} \int_0^{\infty} u^{s-1} P(u) du &= \frac{2^{s/2} \Gamma x (\mu/2+s/2+1/4-l \pm n)}{\Gamma (\mu/2+s/2+1/4-2l)} \cdot \frac{2^{s/2-1/2} \Gamma (v/2-s/2+3/4-2k)}{\Gamma x (v/2-s/2+3/4-k \pm m)} \\ &\quad \cdot \frac{\phi (s)}{\psi (s)} \\ &= \frac{2^s \Gamma x (v/2+s/2+1/4-k \pm m)}{\Gamma (v/2+s/2+1/4-2k)} \cdot \frac{\Gamma x (\mu/2+s/2+1/4-l \pm n)}{\Gamma (\mu/2+s/2+1/4-2l)} \omega (s), \end{aligned}$$

$$\text{where } \omega (s) = \frac{1}{\sqrt{2}} \frac{\Gamma (v/2-s/2+3/4-2k)}{\Gamma x (v/2+s/2+1/4-k \pm m)} \frac{\Gamma (v/2+s/2+1/4-2k)}{\Gamma x (v/2-s/2+3/4-k \pm m)}.$$

Hence $\omega (s)$ satisfies (2.2).

Thus, by Mellin's Inversion formula, we get

$$P(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^s \frac{\Gamma x (v/2+s/2+1/4-k \pm m)}{\Gamma (v/2+s/2+1/4-2k)} \frac{\Gamma x (\mu/2+s/2+1/4-l \pm n)}{\Gamma (\mu/2+s/2+1/4-2l)} \omega (s) x^{-s} ds.$$

Thus the theorem is established.

Corollary:—If $g(x)$ is $R_{\mu}(l+\frac{1}{2}, n)$ and $P(x)$ is given by (6.1), then $f(x)$ is $R_v(k+\frac{1}{2}, m)$.

7. Another rule he has given is:

If $f(x)$ is $R_v(k+\frac{1}{2}, m)$ and

$$F(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma x (\nu/2 + s/2 + 1/4 - k \pm m)}{\Gamma (\nu/2 + s/2 + 1/4 - 2k)} \frac{\Gamma x (\mu/2 - s/2 + 3/4 - l \pm n)}{\Gamma (\mu/2 - s/2 + 3/4 - 2l)} \omega(s) x^{-s} ds; \quad (7.1)$$

where $\omega(s)$ satisfies (2.2),

(7.2)

$$\text{then } g(x) = \frac{1}{x} \int_0^\infty F\left(\frac{y}{x}\right) f(y) dy$$

belongs to $R_\mu(l + \frac{1}{2}, n)$.

The converse theorem is:

If $f(x)$ is $R_\nu(k + \frac{1}{2}, m)$ and $g(x)$ given by (7.2) is $R_\mu(l + \frac{1}{2}, n)$, then $F(x)$ will be given by (7.1).

Another converse theorem is:

If $g(x)$ given by (7.2) is $R_\mu(l + \frac{1}{2}, n)$ and $P(x)$ is given by (7.1), then $f(x)$ is

$R_\nu(k + \frac{1}{2}, m)$.

8. Theorem:—The resultant

(8.1)

$$P(x) = \int_0^\infty f_1(xy) f_2(y) dy$$

of two functions given by

$$f_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s/2} \Gamma x (\nu/2 + 1/4 + s/2 - k \pm m)}{\Gamma (\nu/2 + s/2 + 1/4 - 2k)} \phi(s) x^{-s} ds, \quad (8.2)$$

$$\text{and } f_2(x) = \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{2^{s/2} \Gamma x (\mu/2 + s/2 + 1/4 - l \pm n)}{\Gamma (\mu/2 + s/2 + 1/4 - 2l)} \psi(s) x^{-s} ds, \quad (8.3)$$

where $0 < c < 1$, & $0 < c' < 1$ and $\phi(x)$ & $\psi(s)$ both satisfy (2.2), is a kernel transform-
ing $R_\nu(k + \frac{1}{2}, m)$ into $R_\mu(l + \frac{1}{2}, n)$.

Proof:—Consider the resultant function

$$P(x) = \int_0^\infty f_1(xy) f_2(y) dy$$

$$= \int_0^{\infty} f_2(y) dy \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s/2} \Gamma(x(v/2+s/2+1/4-k\pm m))}{\Gamma(v/2+s/2+1/4-2k)} \phi(s) (xy)^{-s} ds.$$

from (8.2)

Changing the order of integration, which is permissible, we obtain,

$$P(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s/2} \Gamma(x(v/2+s/2+1/4-k\pm m))}{\Gamma(v/2+s/2+1/4-2k)} x^{-s} \phi(s) ds \int_0^{\infty} y^{-s} f_2(y) dy.$$

Now, applying Mellin's Inversion formula to (8.3) and changing s into $1-s$ and also remembering that $\psi(s) = \psi(1-s)$, we get

$$\int_0^{\infty} y^{-s} f_2(y) dy = 2^{1/2-s/2} \frac{\Gamma(x(\mu/2-s/2+3/4-l\pm n))}{\Gamma(\mu/2+3/4-s/2-2l)} \psi(s).$$

$$\text{Hence } P(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{1/2} \Gamma(x(v/2+s/2+1/4-k\pm m))}{\Gamma(v/2+s/2+1/4-2k)} \frac{\Gamma(x(\mu/2-s/2+3/4-l\pm n))}{\Gamma(\mu/2-s/2+3/4-2l)} \phi(s) \psi(s) x^{-s} ds.$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(x(v/2+s/2+1/4-k\pm m))}{\Gamma(v/2+s/2+1/4-2k)} \frac{\Gamma(x(\mu/2-s/2+3/4-l\pm n))}{\Gamma(\mu/2-s/2+3/4-2l)} \omega(s) x^{-s} ds,$$

where $\omega(s) = \sqrt{2} \phi(s) \psi(s)$ and satisfies (2.2).

Thus $P(x)$ is a kernel transforming $R_\nu(k+\frac{1}{2}, m)$ into $R_\mu(l+\frac{1}{2}, n)$.

To justify the change in the order of integration we consider the integral

$$\int_0^{\infty} |f_2(y)| dy \int_{c-i\infty}^{c+i\infty} \left| \frac{2^{s/2} \Gamma(v/2+s/2+1/4-k\pm m)}{\Gamma(v/2+s/2+1/4-2k)} \phi(s) (xy)^{-s} \right| / ds \quad (8.4)$$

On putting $s = c+it$ and using the Asymptotic formula

$$|\Gamma(A+it)| \sim C e^{-\pi/4 |t|} |t|^{A-1/2} \text{ as } |t| \rightarrow \infty,$$

We see that (8.4) is not greater than a constant multiple of

$$x^{-c} \int_0^{\infty} |f_2(y)| y^{-c} dy \int_{-\infty}^{\infty} 2^{c/2} e^{-\pi/4 |t|} |t|^{v/2+c/2-1/4} \phi(c-it) dt,$$

The change in the order of integration will be thus permitted if both the y and t integrals separately exist. They will exist if both the functions $f_1(x)$ and $f_2(x)$ belong to $A(\alpha, a)$, (Mehra Thesis).

Now, the modulus of the integrand in (8.4) is not greater than a constant multiple of $y^{-\sigma} |t|^{v/2 + \sigma/2 - 1/4} e^{-|t|(\alpha - \eta - |\odot|)}$

for $\phi(s) = O(e^{(\pi/4 - \alpha + \eta)|t|})$

for every positive $\eta < \alpha$ and uniformly in any strip interior to $a < \sigma < 1-a$; where ϵ is any value of σ in the above strip. Thus, the integral converges uniformly in any domain of x interior to the angle α , and in any smaller angle, is $O(|x|^{-\epsilon})$. Since ϵ may be taken as near $(1-a)$ or a as we please, $f_1(x)$ belongs to $A(\alpha, a)$. Similarly $f_2(x)$ also belongs to $A(\alpha, a)$.

In order to illustrate the above theorem let us take examples:

Example 1. Let $f_1(x) = x^{-\frac{1}{2}}$, which is $R_{\mu}(l+\frac{1}{2}, n)$,

and $f_2(x) = x^{v-2k-\frac{1}{2}} e^{-\frac{1}{2}x^2} W_{k+\frac{1}{2}, m}\left(\frac{x^2}{2}\right)$

which is $R_{\nu}(k+\frac{1}{2}, m)$.

Then the resultant

$$\begin{aligned} P(x) &= \int_0^{\infty} f_1(xy) f_2(y) dy \\ &= \int_0^{\infty} (xy)^{-\frac{1}{2}} y^{v-2k-\frac{1}{2}} e^{-\frac{1}{2}y^2} W_{k+\frac{1}{2}, m}\left(\frac{1}{2}y^2\right) dy \end{aligned}$$

By a formula given by Goldstein, we have

$$P(x) = 2^{v/2-k-1} \frac{\Gamma x(v/2+1/2-k \pm m)}{\Gamma(v/2+1/2-2k)} x^{-\frac{1}{2}},$$

is a kernel transforming $R_{\mu}(l+\frac{1}{2}, n)$ to $R_{\nu}(k+\frac{1}{2}, m)$.

Corollary:—On putting $\mu = \nu$ and $k = -m$, we obtain a particular case of a result due to Dr. Brij Mohan⁷ which is "The resultant of two R_{ν} function is a resultant of the above type".

9. The resultant

$$K(x) = \int_0^\infty P_1(y) P_2(xy) dy, \quad (9.1)$$

of the two kernels

$$P_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma x (\nu/2+s/2+1/4-k\pm m) \Gamma x (\mu/2-s/2+3/4-l\pm n)}{\Gamma (\nu/2+s/2+1/4-2k) \Gamma (\mu/2+s/2+3/4-2l)} \omega_1(s) x^{-s} ds \quad (9.2)$$

$$\text{and } P_2(x) = \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{2^s \Gamma x (\nu/2+s/2+1/4-k\pm m) \Gamma x (\mu'/2+s/2+1/4-l'\pm n')}{\Gamma (\nu/2+s/2+1/4-2k) \Gamma (\mu'/2+s/2+1/4-2l')} \omega_2(s) x^{-s} ds, \quad (9.3)$$

where $0 < c < 1$, $0 < c' < 1$,

and $\omega_1(s)$ and $\omega_2(s)$ both satisfy (2.2).

The proof follows from the theorem given above.

Example:—Let us take in (9.2)

$$\omega(s) = \frac{\prod_{j=3}^r \pi \Gamma(1/4+b_j+s/2)}{r-1} \frac{\prod_{j=3}^r \pi \Gamma(3/4-a_j-s/2)}{r-1} \frac{2 \prod_{j=q+1}^p \pi \Gamma(3/4-b_j-s/2)}{p} \frac{\prod_{j=q+1}^p \pi \Gamma(1/4+a_j+s/2)}{p},$$

which satisfies the relation (2.2), if

$$a_j + b_j = 0; j = 3, 4, \dots, r-1.$$

On replacing $-s/2$ by $t+1/4$, we obtain

$$P_1(x) = A \sqrt{x} G_{r,r}^{q,q} \left(x^2 \left| \begin{matrix} -\mu/2+l-n, -\mu/2+l+n, -3b_q, -q+1, b_{r-1}, \nu/2-2k \\ \nu/2-k+m, \nu/2-k-m, 3b_q, q+1, b_{r-1}, -\mu/2+2l \end{matrix} \right. \right),$$

where $2q > r > q$,

as a kernel transforming $R_\nu(k+\frac{1}{2}, m)$ into $R_\mu(l+\frac{1}{2}, n)$.

Again, putting

$$\omega_2(x) = \frac{\prod_{j=1}^n \pi \Gamma(1/4+b_j+s/2)}{\sqrt{2} \prod_{j=n+1}^p \pi \Gamma(3/4-b_j-s/2)} \frac{\prod_{j=1}^n \pi \Gamma(3/4-a_j-s/2)}{\prod_{j=n+1}^p \pi \Gamma(1/4+a_j+s/2)},$$

which satisfies the functional relation (2.2), if

$$a_j + b_j = 0, j = 1, 2, \dots, p;$$

and changing $-s/2$ to $t+1/4$ in (9.3), we obtain

$$P_2(x) = A \sqrt{x} G_{p+2, p+4}^{n+4, n} \left(\frac{x^{3/4}}{1} \left| \begin{matrix} -1^{b_p}, v/2-2k, \mu'/2-2l' \\ v/2-k+m, v/2-k-m, \mu'/2-l'+n', \mu'/2-l'-n', 1^{b_p} \end{matrix} \right. \right),$$

where $2n+1 > p > n$.

as a kernel transforming $R_\nu(k+\frac{1}{2}, m)$ into $R_{\mu'}(l'+\frac{1}{2}, n')$.

Hence the resultant kernel is given by

$$K(x) = A \int_0^\infty \sqrt{y} G_{r, r}^{q, q} \left(y^2 \left| \begin{matrix} -\mu/2+l-n, -\mu/2+l+n, -3^b q, -q+1^{b_r-1}, v/2-2k \\ v/2-k+m, v/2-k-m, 3^b q, q+1^{b_r-1}, -\mu/2+2l \end{matrix} \right. \right) \\ \times \sqrt{xy} G_{p+2, p+4}^{n+4, n} \left(\frac{x^2 y^2}{1} \left| \begin{matrix} -1^{b_p}, v/2-2k, \mu'/2-2l' \\ v/2-k+m, v/2-k-m, \mu'/2-l'+n', \mu'/2-l'-n', 1^{b_p} \end{matrix} \right. \right) dy$$

Putting $y^2 = u$ and using (Meijer 6), we obtain

$$K(x) = A \sqrt{x} G_{p+r+2, p+r+4}^{n+q+4, n+q} \left(\frac{x^2}{4} \left| \begin{matrix} -v/2+k-m, -v/2-k+m, -3^b q, -1^{b_p} \\ -\mu/2-l+n, -\mu/2-l-n, 3^b q, v/2-k-m, \\ v/2-2k, \mu'/2-2l', -q+1^{b_r+1}, \mu/2-2l; \\ v/2-k+m, \mu'/2-l'-n', \frac{\mu'}{2}-l'+n', 1^{b_p}, q+1^{b_r-1}, -v/2+2k \end{matrix} \right. \right)$$

where $|\arg x^{3/4}| < (2n-p+1)\pi$, and $p+r < 2n+2q+1$, as a kernel transforming $R_\mu(l+\frac{1}{2}, n)$ into $R_{\mu'}(l'+\frac{1}{2}, n')$.

10. *Theorem*:—The resultant

$$K(x) = \int_0^\infty P_2(xy) P_2(y) dy \quad (10.1)$$

of the two kernels of the form:

$$P_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^s \Gamma x(v/2+1/4+s/2-k\pm m) \Gamma x(\mu/2+s/2+1/4-l\pm n)}{\Gamma(v/2+s/2+1/4-2k) \Gamma(\mu/2+s/2+1/4-2l)} \omega_1(s) x^{-s} ds. \quad (10.2)$$

$$\text{and } P_2(x) = \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{2^s \Gamma x(v/2+s/2+1/4-k\pm m) \Gamma x(\mu'/2+s/2+1/4-l'\pm m')}{\Gamma(v/2+s/2+1/4-2k) \Gamma(\mu'/2+s/2+1/4-2l')} \omega_2(s) x^{-s} ds. \quad (10.3)$$

where $0 < c < 1$, $0 < c' < 1$

and $\omega_1(s)$ and $\omega_2(s)$ both satisfy (2.2).

is a kernel transforming $R_\mu(l+\frac{1}{2}, n)$ into $R_{\mu'}(l'+\frac{1}{2}, n')$.

Proof follows from theorem 8.

Example:—Let us take $\omega_1(s)$ in (10.2) same as $\omega_2(s)$ in (9.3).

Then

$$P_1(x) = A \sqrt{x} G_{p+2, p+4}^{n+4, n} \left(\frac{x^2}{4} \left| \begin{matrix} -1 \frac{b}{n}, -n+1 \frac{b}{p}, v/2-2k, \mu/2-2l \\ v/2-k+m, v/2-k-m, \mu'/2-l'+n', \mu'/2-l'-n', 1 \frac{b}{n}; n+1 \frac{b}{p} \end{matrix} \right. \right),$$

where $2n+1 > p > n$.

is a kernel transforming $R_v(k+\frac{1}{2}, m)$ into $R_\mu(l+\frac{1}{2}, n)$. Again taking $\omega_2(s)$ same as $\omega_1(s)$ in (9.3) with n and p replaced by q and r respectively, we get

$$P_2(x) = A \sqrt{x} G_{r+2, r+4}^{q+4, q} \left(\frac{x^2}{4} \left| \begin{matrix} -1 \frac{b}{q}, q+1 \frac{b}{r}, v/2-2k, \mu'/2-2l' \\ v/2-k+m, v/2-k-m, \mu'/2-l'+n', \mu'/2-l'-n', 1 \frac{b}{q}; q+1 \frac{b}{r} \end{matrix} \right. \right),$$

where $2q+1 > r > q$

as a kernel transforming $R_v(k+\frac{1}{2}, m)$ into $R_{\mu'}(l'+\frac{1}{2}, n')$

Hence the resultant kernel will be given by

$$K(x) = A \sqrt{x} G_{p+r+6, p+r+6}^{n+q+4, n+q+4} \left(x^2 \left| \begin{matrix} -v/2+k-m, -v/2+k+m, -u'/2+l'-n', -\mu'/2+l'+n', -1 \frac{b}{q}, -1 \frac{b}{p}, \\ 1 \frac{b}{q}, v/2-k+m, v/2-k-m, \mu'/2-l+n, \mu'/2-l-n, 1 \frac{b}{p}, q+1 \frac{b}{r}, \\ v/2-2k, \mu/2-2l-q+1 \frac{b}{r}, \\ -v/2+2k, -\mu'/2+2l' \end{matrix} \right. \right),$$

where $|\arg x^2| < (2n-p+1)\pi$ and $p+r < 2n+2q+2$,

which is a kernel transforming $R_\mu(l+\frac{1}{2}, n)$ into $R_{\mu'}(l'+\frac{1}{2}, n')$

Theorems converse to the above theorems as for the previous ones may also be given.

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CHEMICAL EXAMINATION OF THE PECTIN FROM THE FRUIT PULP
OF FERONIA ELEPHANTUM : PART I. THE STRUCTURE OF
GALACTANS AND ARABANS ACCOMPANYING THE
POLYGALACTURONIC ACID

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ABSTRACT

The pectin isolated by extracting the fruit pulp of *Feronia elephantum* Correa gave the following results on preliminary examination: ash 1.68%; methoxyl 8.61%, methyl ester 11.7% acetyl ester 2.28%; uronide 60.3%, and calcium pectate 79.40%. Hydrolysis with 2N sulphuric acid and its chromatographic examination indicated the presence of galactose, arabinose and galacturonic acid. Methylation of pectin with dimethyl sulphate and alkali followed by examination of the hydrolysate showed the presence of three methylated galactoses, viz. 3,4-di-O-methyl-D-galactose; 2,3,6-tri-O-methyl-D-galactose and 2,3,4,6-tetra-O-methyl-D-galactose and three methylated arabinoses i.e. 2-O-methyl-L-arabinose, 2,3-di-O-methyl-L-arabinose and 2,3,5-tri-O-methyl-L-arabinose thus indicating the carbon atoms free for linkage. On the basis of these investigations, structures have been assigned for the galactans and arabans which are present with the polygalacturonic acid molecule of pectin.

Wood apple (*Feronia elephantum*) belonging to the natural order Rutaceae is a deciduous tree found throughout India. The components from various parts of the tree are reputed for their pharmacological importance^{1,2} and have been investigated from time to time^{3,4,5}.

The present investigation deals with the examination of the fruit pulp of wood apple for its pectin content and determination of the structure of the galactans and arabans accompanying the polygalacturonic acid.

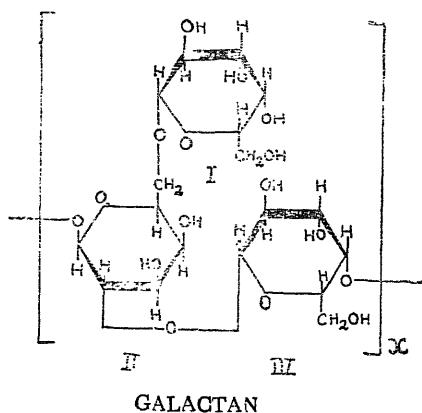
From the fruit pulp a pectin has been isolated by extraction with dilute hydrochloric acid, followed by precipitation by ethanol. The crude pectin has been purified by repeating the above procedure till a sample of pectin free from nitrogen and with minimum ash content was obtained. The galactans were left in traces and the araban content was considerably reduced by this method of purification. The purified pectin was used for preliminary analysis whereas the sample of pectin which had been partially purified was used for methylation studies.

On preliminary examination pectin gave the following results, ash 1.68%; methoxyl 8.61%; methyl ester 11.7%; acetyl ester 2.28%; uronide 60.3%; calcium pectate 79.40%. Complete hydrolysis of the pectin with 2N sulphuric acid and the paper chromatographic examination of the hydrolysate revealed the presence of galactose, arabinose, and galacturonic acid. Since the amounts of galactose and arabinose present in the pectin have been found to diminish gradually with repeated purifications, it indicates that probably galactose and arabinose are not a part of the main structure of pectin but are present as galactans and arabans along with the polygalacturonic acid molecule. In view of the high uronide

content (60.3%), the main building unit appears to be polygalacturonic acid, the galactans and arabans being coprecipitated with it.

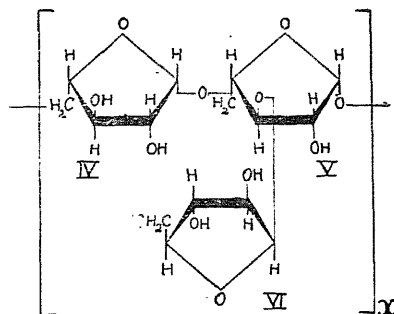
The highly purified pectin resisted methylation with dimethyl sulphate and alkali^{6,7} and with Purdie's reagents.⁸ When the crude pectin was repeatedly methylated at 30°C with dimethyl sulphate and alkali,⁷ a methylated product was obtained which on hydrolysis followed by paper chromatographic examination revealed the presence of the following methylated sugars which were identified by comparing their R_g values (distance moved by the methylated sugar divided by the distance moved by 2,3,4,6-tetra-O-methyl-D-glucose) with those given in literature.^{9,10} The presence of 3,4-di-O-methyl-D-galactose; 2,3,6-tri-O-methyl-D-galactose, 2,3,4,6-tetra-O-methyl-D-galactose, 2-O-methyl-L-arabinose, 2,3-di-O-methyl-L-arabinose, and 2,3,5-tri-O-methyl-arabinose has thus been established. Since no methylated galacturonic acid was obtained it is clear that the uronic acid part of the molecule resists methylation whereas the arabans and galactans are methylated.

Corresponding to the galactan three different methylated galactoses viz. (1) 3,4-di-O-methyl-D-galactose (2) 2,3,6-tri-O-methyl-D-galactose and (3) 2,3,4,6-tetra-O-methyl-D-galactose have been identified. Since 3,4-di-O-methyl-D-galactose denoted by II in the proposed structure, has been identified, this must be linked through its C₂ and C₆ to III and I and the entire unit repeated at C₁. The presence of 2,3,6-tri-O-methyl-D-galactose indicates that this molecule III in the structure must be linked through its C₁ to II and the unit repeated at C₄. 2,3,4,6-Tetra-O-methyl-D-galactose has also been identified in the hydrolysate and since only C₁ of this molecule is free, it is evident that this I must form a side chain in the repeating unit and must be attached through its C₁ to C₆ of II. This is also in keeping with the fact that all three of these methylated galactoses have been found to be in equimolecular proportions.



Further it will be noted that the following methylated arabinose derivatives have been identified in the hydrolysate (1) 2-O-methyl-L-arabinose (2) 2,3-di-O-

methyl-L-arabinose and (3) 2,3,5-tri-O-methyl-L-arabinose showing thereby that the araban may have the following structure.



ARABAN

Since 2-O-methyl-L-arabinose has been identified, this V must be linked to IV and VI through its C₅ and C₃ and the unit repeated at C₁. Also the presence of 2,3-di-O-methyl-L-arabinose shows that this IV must be linked through its C₁ to V and to the rest of the chain through C₅. 2,3,5-Tri-O-methyl-L-arabinose has also been identified and since this has only C₁ free it is evident that this VI forms a side chain in the repeating unit and must be attached through its C₁ to C₃ of V in the proposed structure. The methylated arabinose derivatives have also been found to be equimolecular in proportion.

EXPERIMENTAL

Unless otherwise stated, specific rotations were measured in aqueous solutions ; paper chromatographic separations were carried out on Whatman No. 1 paper using the following solvents (A) butanol : ethanol : water (40:10:50)¹¹ (B) butanol : ethanol : ammonia : water (40:10:1:49) (C) butanol : acetic acid : water (5:1:4) and (D) butanone water azeotrope¹². The spray reagents used were aniline hydrogen phthalate, silver nitrate in acetone and ethanolic sodium hydroxide, naphthoresorcinol and trichloroacetic acid.

Isolation and Purification of the Pectin—Wood apple was crushed and refluxed with 90% ethanol to remove the alcohol soluble substances. The pulp was filtered and air dried. The pulp could now be stored without fungal attack.

The dry pulp was refluxed with 0.013N hydrochloric acid on a water bath for three to four hours, using 100ml acid for every 10-15 gm of the pulp.¹³ The extract was filtered through cloth and centrifuged to remove suspended particles. The centrifuged solution was refiltered through a B₃ sintered funnel. 90% Ethanol was added with continuous stirring to the filtrate till the final concentration of ethanol in the solution reached 50-60%. Pectin precipitated out as a flocculent jelly. The mixture was allowed to stand as such for several hours. The crude pectin was filtered, washed 2-3 times with ethanol and finally with ether and dried in vacuum. The pectin was purified by reprecipitations by dissolving it in warm water and pouring ethanol in it. This procedure was repeated several times till it gave a

negative test for nitrogen and the ash content was reduced to a minimum. The purified pectin had a specific rotation $(\alpha)_D^{30} -181.8^\circ$.

PRELIMINARY ANALYSIS OF PECTIN

Ash Content—About 0.05 gm of the purified pectin was taken in a previously weighed platinum crucible and incinerated. After heating to a constant weight the percentage of ash was determined. Found ash 1.68%.

Methoxyl group—The methoxyl group was determined by heating the pectin with hydriodic acid and estimating the methyl iodide so formed by the method of Belcher et al.¹⁴ Found methoxyl 11.07%.

Uronide content—0.025 gm of pectin was hydrolysed with 19% M. A. R. hydrochloric acid. The carbon dioxide so formed was absorbed in 5% sodium hydroxide and estimated volumetrically in accordance to the method of Barker, Foster, Siddiqui and Stacey.¹⁵ Found uronide content 60.3%

Methyl ester by saponification—0.1 gm of pectin was weighed in a flask and dissolved in 40 ml water. This was neutralised with 0.1 N sodium hydroxide, using a mixture of 4% solutions of phenol red, cresol red, bromothymol blue and distilled water in the ratio of 3:1:1:1 as indicator.¹⁶ Now 10 ml of 0.05N sodium hydroxide were added and the flask allowed to stand for one half hour. The excess of alkali was then titrated against 0.1N hydrochloric acid. Blanks were also run. Found methyl ester 11.7%.

Calcium pectate—The method of Carré and Haynes¹⁷ as described by Kertesz,¹⁸ was followed. Found calcium pectate 79.40%.

Hydrolysis and identification of the sugars in the hydrolysate—0.05 gm of pectin was hydrolysed with 2N sulphuric acid on a water bath for 24 hr. The solution was neutralised with barium carbonate filtered and concentrated under reduced pressure. Paper chromatograms of this solution were run in solvents (B) and (C) and each sprayed with a different reagent. The R_f values were measured and compared with those given in literature.

Solvent B		Solvent C		Sugar Identified
R _f found	R _f in lit.	R _f found	R _f in lit.	
0.085	0.09	0.155	0.16	Galactose
0.024	...	0.13	0.14	Galacturonic acid
0.14	0.145	0.20	0.21	Arabinose

Spots of galactose, arabinose, galacturonic acid and the hydrolysate were applied on another chromatogram which was run in solvent (A). After spraying with aniline hydrogen phthalate three spots were seen which corresponded with these reference sugars. Glucose and galactose were further differentiated by forming their benzylamine derivatives¹⁹ by superimposing spots of a 10% methanolic solution of benzylamine on spots of the hydrolysate, glucose and galactose. The paper was heated at 85° for five minutes and developed in solvent (B) and sprayed with a

0.01% solution of ninhydrin in butanol. The hydrolysate gave a spot corresponding to the benzylamine derivative of galactose.

Quantitative estimation of the monosaccharides in the hydrolysate of the purified pectin—A weighed amount of the purified pectin was hydrolysed by 1.5N sulphuric acid, neutralised with barium carbonate and filtered.²⁰ The filtrate was made up to 25 ml. A known volume of this solution was applied on a paper chromatogram along with guide spots. After developing in solvent (B), strips corresponding to the sugars were cut with the help of guide spots and eluted. The elutes were separately oxidised by periodate⁶ and the quantity of monosaccharides determined. Blanks were also run simultaneously. Since galactose was not present in the highly purified pectin the paper strip between the uronic acid and arabinose was eluted and oxidised. No periodate consumption took place in this elute. Thus it was inferred that galactose was not present in this sample.

Arabinose was determined and found to be 38.92%.

Methylation—Pectin (3 gm) was dissolved in water (20 ml), dimethyl sulphate (30 ml) and 40% sodium hydroxide (60 ml) were added with constant stirring during 8 hours. The temperature was maintained at 30°, during addition. Methylation was also carried out at 0° but the methylated product was not obtained by this method. After two repetitions of the above procedure at 30° the solution was concentrated to a thick syrup which was dissolved in acetone. This was remethylated thrice. The final concentrated syrup was extracted continuously with chloroform in a liquid-liquid extractor (downward displacement type) for 20 hr. The chloroform extract was dried over anhydrous sodium sulphate and distilled under reduced pressure to remove chloroform. The methylated product was a thick syrup ($\alpha_D^{30} + 91.8$ in chloroform).

Hydrolysis and identification of the methylated products—The methylated pectin ($\alpha_D^{30} + 91.8$ in CHCl_3) was hydrolysed with 1 N sulphuric acid and methanol at 100° for 25 hr. After neutralising the hydrolysate with barium carbonate the filtrate was concentrated in vacuo and examined paper chromatographically. The chromatograms were developed in solvents (A) and (D) and sprayed with aniline hydrogen phthalate. The R_f and R_g values were calculated in each case, compared with those given in literature^{9,10} and the sugars identified.

Solvent (A)		Solvent (B)		Sugar Identified
R_g found	R_g given	R_f found	R_f given	
0.33	0.32	3,4-di-O-methyl-D-galactose
0.37	0.38	0.145	0.14	2-O-methyl-L-arabinose
0.63	0.64	2,3-di-O-methyl-L-arabinose
0.72	0.71	0.47	0.48	2,3,6-tri-O-methyl-D-galactose
0.87	0.88	0.67	0.68	2,3,4,6-tetra-O-methyl-D-galactose
0.94	0.95	0.83	0.82	2,3,5-tri-O-methyl-L-arabinose

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CERTAIN KERNEL FUNCTIONS

By

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ABSTRACT

In this paper some theorems on kernels with their particular cases connecting different classes of self-reciprocal functions, have been given and illustrated by means of certain examples.

1. Recently Roop Narain [4] has given a generalisation of the well-known Hankel transform, namely

$$g(x) = \int_0^{\infty} \sqrt{xy} J_{\nu}(xy) f(y) dy, \quad (1.1)$$

by means of the integral equation

$$g(x) = \frac{1}{2\nu} \int_0^{\infty} (xy)^{\nu+\frac{1}{2}} \chi_{\nu, k, m} \left(\frac{1}{2} x^2 y^2 \right) f(y) dy, \quad (1.2)$$

where $\chi_{\nu, k, m}(x)$ is a function, expressed in its simplest form in terms of G-function as :

$$\chi_{\nu, k, m}(x) = x^{-\nu} G_{24}^{21} \left(x \left/ \begin{matrix} k+m-\frac{1}{2}, \nu-k+m+\frac{1}{2} \\ \nu, \nu+2m, 0, -2m \end{matrix} \right. \right), \quad (1.3)$$

It is easy to see that

$$\chi_{\nu, k, m}(x) = O(x^{m \pm m}),$$

for small x , and $\chi_{\nu, k, m}(x) = O(x^{-\lambda})$,

for large x , where $\lambda = \frac{1}{2}(\nu + \frac{1}{2}), \frac{1}{2} + k - m$ or $\nu + 3/2 + m - k$.

In particular, when $k+m = \frac{1}{2}$, (1.2) reduces to (1.1). The function $g(x)$ given by (1.2) is known as the $\chi_{\nu, k, m}$ -transform of $f(x)$. If $g(x) = f(x)$, then $f(x)$ is said to be self-reciprocal in the $\chi_{\nu, k, m}$ -transform and is written as $R_{\nu}(k, m)$.

The necessary and sufficient condition for a function $f(x)$ of $A(\alpha, a)$ to be $R_{\nu}(k, m)$ is that it should be of the form :

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s/2} \Gamma_x(\nu/2 + s/2 + \frac{1}{2} + m \pm m)}{\Gamma(\nu/2 + s/2 + \frac{3}{2} + m - k)} \phi(s) x^{-s} ds, \quad (1.4)$$

where $\phi(s)$ is regular and satisfies the equation

$$\phi(s) = \phi(1-s), \quad (1.5)$$

in the strip $a < \sigma < 1-a$,

$$(1.6)$$

A number of kernels connecting different classes of self reciprocal functions for this transform have been obtained by Roop Narain and by R. K. Saxena. In this note the theory of kernel functions has been further developed and new kernels have been introduced.

2. *Theorem 1.* If $f(x)$ is $R_{\mu(l,n)}$ and $k(x)$ is $R_{\nu}(k,m)$ then

$$\text{the function } g(x) = \int_0^{\infty} P(xy) dy \int_0^{\infty} k(u) f(uy) du, \quad (2.1)$$

is $R_{\lambda}(p,q)$, provided that

$$P(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{s/2} \frac{\Gamma_x(\mu/2+s/2+\frac{1}{4}+n\pm n) \Gamma_x(\nu/2+\frac{3}{4}-s/2+m\pm m)}{\Gamma(\mu/2+s/2+\frac{3}{4}+n-l) \Gamma(\nu/2-s/2+5/4+m-k)} \frac{\Gamma_x(\lambda/2+s/2+\frac{1}{4}+q\pm q)}{\Gamma(\lambda/2+s/2+\frac{3}{4}+q-p)} \omega(s) x^{-s} ds, \quad (2.2)$$

where $\omega(s)$ satisfies (1.5)

$$\text{Proof. Let us take } Q(y) = \int_0^{\infty} k(u) f(uy) du. \quad (2.3)$$

It can be easily seen that

$$Q(y) = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \frac{\Gamma_x(\mu/2+s/2+\frac{1}{4}+n\pm n) \Gamma_x(\nu/2-s/2+\frac{3}{4}+m\pm m)}{\Gamma(\mu/2+s/2+\frac{3}{4}+n-l) \Gamma(\nu/2-s/2+5/4+m-k)} \psi(s) y^{-s} ds, \quad (2.4)$$

where $\psi(s)$ satisfies (1.5)

Hence with the help of (2.4) we obtain

$$g(x) = \frac{1}{2\pi i} \int_0^{\infty} P(xy) dy \int_{k-i\infty}^{k+i\infty} \frac{\Gamma_x(\mu/2+s/2+\frac{1}{4}+n\pm n) \Gamma_x(\nu/2-s/2+\frac{3}{4}+m\pm m)}{\Gamma(\mu/2+s/2+\frac{3}{4}+n-l) \Gamma(\nu/2-s/2+5/4+m-k)} \psi(s) x^{-s} ds,$$

$$= \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \frac{\Gamma_x(\mu/2+s/2+\frac{1}{4}+n\pm n) \Gamma_x(v/2-s/2+\frac{3}{4}+m\pm m)}{\Gamma(\mu/2+s/2+\frac{3}{4}+n-l) \Gamma(v/2-s/2+5/4+m-k)} \psi(s) ds \int_0^\infty y^{-s} P(xy) dy,$$

provided that the inversion of the order of integration is justified.

Therefore,

$$g(x) = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \frac{\Gamma_x(\mu/2+s/2+\frac{1}{4}+n\pm n) \Gamma_x(v/2-s/2+\frac{3}{4}+m\pm m)}{\Gamma(\mu/2+s/2+\frac{3}{4}+n-l) \Gamma(v/2-s/2+5/4+m-k)} \psi(s) x^s - 1 ds \int_0^\infty P(u) u^{-s} du,$$

where $u=xy$ and $du = x dy$.

Now, by Mellin's Inversion formula [2,] we get

$$\int_0^\infty P(u) u^s - 1 du = \frac{2^{s/2} \Gamma_x(\mu/2+s/2+\frac{1}{4}+n\pm n) \Gamma_x(v/2-s/2+\frac{3}{4}+m\pm m)}{\Gamma(\mu/2+s/2+\frac{3}{4}+n-l) \Gamma(v/2-s/2+5/4+m-k)} \frac{\Gamma_x(\lambda/2-s/2+\frac{3}{4}+q\pm q)}{\Gamma(\lambda/2+s/2+\frac{3}{4}+q-p)} \omega(s)$$

Again, writing $1-s$ for s , we get

$$\int_0^\infty P(u) u^{-s} du = \frac{2^{\frac{1}{2}-s/2} \Gamma_x(\mu/2-s/2+\frac{3}{4}+n\pm n) \Gamma_x(v/2+s/2+\frac{1}{4}+m\pm m)}{\Gamma(\mu/2-s/2+5/4+n-l) \Gamma(v/2+s/2+\frac{3}{4}+m-k)} \frac{\Gamma_x(\lambda/2-s/2+\frac{3}{4}+q\pm q)}{\Gamma(\lambda/2-s/2+5/4+q-p)} \omega(s)$$

since $\omega(s)$ satisfies (1.5).

Hence,

$$g(x) = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \frac{2^{\frac{1}{2}-s/2} \Gamma_x(\mu/2-s/2+\frac{3}{4}+n\pm n) \Gamma_x(v/2+s/2+\frac{1}{4}+m\pm m)}{\Gamma(\mu/2-s/2+5/4+n-l) \Gamma(v/2+s/2+\frac{3}{4}+m-k)} \frac{\Gamma_x(\lambda/2-s/2+\frac{3}{4}+q\pm q) \Gamma_x(\mu/2+s/2+\frac{1}{4}+n\pm n) \Gamma_x(v/2-s/2+\frac{3}{4}+m\pm m)}{\Gamma(\lambda/2-s/2+5/4+q-p) \Gamma(\mu/2+s/2+\frac{3}{4}+n-l) \Gamma(v/2-s/2+5/4+m-k)} \times x^s - 1 \omega(s) + (s) ds.$$

On changing s into $(1-s)$, we obtain

$$g(x) = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \frac{2^{s/2} \Gamma_x(\mu/2 + s/2 + \frac{1}{4} + n \pm n) \Gamma_x(\nu/2 - s/2 + \frac{3}{4} + m \pm m)}{\Gamma(\mu/2 + s/2 + \frac{3}{4} + n - l) \Gamma(\nu/2 - s/2 + 5/4 + m - k)} \\ \frac{\Gamma_x(\lambda/2 + s/2 + \frac{1}{4} + q \pm q) \Gamma_x(\mu/2 - s/2 + \frac{3}{4} + n \pm n) \Gamma_x(\nu/2 + s/2 + \frac{1}{4} + m \pm m)}{\Gamma(\lambda/2 + s/2 + \frac{3}{4} + q - p) \Gamma(\mu/2 - s/2 + 5/4 + n - l) \Gamma(\nu/2 + s/2 + \frac{3}{4} + m - k)} \\ \times \omega(s) + (s) x^{-s} ds, \\ = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \frac{2^{s/2} \Gamma_x(\lambda/2 + s/2 + \frac{1}{4} + q \pm q)}{\Gamma(\lambda/2 + s/2 + \frac{3}{4} + q - p)} \phi(s) x^{-s} ds,$$

where

$$\phi(s) = \frac{\Gamma_x(\mu/2 + s/2 + \frac{1}{4} + n \pm n) \Gamma_x(\mu/2 - s/2 + \frac{3}{4} + n \pm n) \Gamma_x(\nu/2 + s/2 + \frac{1}{4} + m \pm m)}{\Gamma(\mu/2 + s/2 + \frac{3}{4} + n - l) \Gamma(\mu/2 - s/2 + 5/4 + n - l) \Gamma(\nu/2 + s/2 + \frac{3}{4} + m - k)} \\ \frac{\Gamma_x(\nu/2 - s/2 + \frac{3}{4} + m \pm m)}{\Gamma(\nu/2 - s/2 + 5/4 + m - k)} \omega(s) + (s),$$

and $\phi(s)$ satisfies (1.5).

Thus it follows from the definition that $g(x)$ is $R_{\lambda}(p, q)$.

Particular Cases :—

Case (i) : If $k+m = \frac{1}{2}$, $l+n = \frac{1}{2}$, $p+q = \frac{1}{2}$, then we have well-known theorem due to Dr. Brij Mohan, which is :

If $f(x)$ is R_{μ} and $k(x)$ is R_{ν} , then the function

$$g(x) = \int_0^{\infty} P(xy) dy \int_0^{\infty} k(z) f(yz) dz,$$

is R_{λ} , provided that

$$P(x) = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} 2^{s/2} \Gamma(\mu/2 + s/2 + \frac{1}{4}) \Gamma(\nu/2 - s/2 + \frac{3}{4}) \Gamma(\lambda/2 + s/2 + \frac{1}{4}) \\ \phi(s) x^{-s} ds,$$

where $\phi(s)$ satisfies (1.5).

Case (ii) : If we take $\mu = \nu$, $m = n$ and $k = l$, we get another theorem :

If $f(x) = k(x)$ is $R_{\nu}(k, m)$

$$\text{and } g(x) = \int_0^{\infty} P(xy) dy \int_0^{\infty} k(u) f(xu) dv.$$

is $R_{\lambda}(p, q)$ provided that

$$P(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s/2} \Gamma_x(\lambda/2 + s/2 + \frac{1}{4} + q \pm q)}{\Gamma(\lambda/2 + s/2 + \frac{3}{4} + q - p)} \phi(s) x^{-s} ds,$$

i.e. $P(x)$ is R_{λ} .

Also, it may be shown easily, that if $k(x)$ and $f(x)$ are both $R_{\nu}(k, m)$, then

$$\int_0^{\infty} k(u) f(uy) du = \int_0^{\infty} k(uy) f(u) du,$$

Hence, we arrive at the following corollary :

If $k(x)$ and $f(x)$ are both $R_{\nu}(k, m)$ and $P(x)$ is

$R_{\lambda}(p, q)$, then the function

$$\begin{aligned} g(x) &= \int_0^{\infty} P(xy) dy \int_0^{\infty} k(u) f(uy) du \\ &= \int_0^{\infty} P(xy) dy \int_0^{\infty} k(uy) f(u) du. \end{aligned}$$

is $R_{\lambda}(p, q)$.

Example 1. To illustrate the theorem, let us take

$$f(x) = x^{\mu+2n-\frac{1}{2}} e^{-\frac{1}{4}x^2} W_{l,n} \left(\frac{x^2}{2} \right),$$

which is a $R_{\mu}(l, n)$ function

$$\text{and } k(x) = x^{\nu+2m-\frac{1}{2}} e^{-\frac{1}{4}x^2} W_{k,m} \left(\frac{x^2}{2} \right),$$

which is a $R_{\nu}(k, m)$ function.

$$\text{Then } g(x) = \int_0^{\infty} P(xy) dy \int_0^{\infty} k(u) f(xu) du$$

will be $R_{\lambda}(p, q)$,

$$\int_0^{\infty} k(x) f(xy) dx$$

$$= \int_0^{\infty} y^{\mu+2n-\frac{1}{2}} x^{\mu+\nu+2m+2n-1} e^{-\frac{1}{2}(1+y^2)x^2} W_{k,m}\left(\frac{x^2}{2}\right) W_{l,n}\left(\frac{x^2 y^2}{2}\right) dx,$$

On putting $x^2 = 2u$, $xdx = du$ we get

$$\int_0^{\infty} k(x) f(xy) dx = 2^{\mu/2+\nu/2+m+n-1} y^{\mu+2n-\frac{1}{2}} \int_0^{\infty} u^{\mu/2+\nu/2+m+n-1} e^{-\mu/2(1+y^2)u} W_{k,m}(u) W_{l,n}(uy^2) du,$$

Now by a well-known integral [1, p. 411]

$$\begin{aligned} & \int_0^{\infty} x^{\rho-1} \exp\left[-\frac{1}{2}x(\alpha+\beta)\right] W_{\lambda,\nu}(\beta x) W_{k,\mu}(\alpha x) dx \\ &= \beta^{-\rho} G_{33}^{22} \left(\frac{\beta}{\alpha} \left/ \begin{matrix} \frac{1}{2}+\mu, \frac{1}{2}-\nu, 1-\lambda+\rho \\ \frac{1}{2}+\nu+\rho, \frac{1}{2}-\nu+\rho, k \end{matrix} \right. \right) R(\alpha+\beta) > 0; R(\mu)+R(\nu) < R(\rho)+1. \end{aligned}$$

we get

$$\begin{aligned} \int_0^{\infty} k(x) f(xy) dx &= 2^{\mu/2+\nu/2+m+n-1} y^{-\nu-2m-\frac{1}{2}} \\ & G_{33}^{22} \left(y^2 \left/ \begin{matrix} \frac{1}{2}+m, \frac{1}{2}-n, 1-l+\mu/2+\nu/2+m+n \\ \frac{1}{2}+\mu/2+\nu/2+m+n \pm n, k \end{matrix} \right. \right), \end{aligned}$$

and let $\rho(x) = x^{-\frac{1}{2}}$ which is a $R_{\lambda}(p,q)$ function.

$$\begin{aligned} \text{Then } & \int_0^{\infty} \rho(xy) dy \int_0^{\infty} k(x) f(xy) dx \\ &= 2^{\mu/2+\nu/2+m+n-1} \int_0^{\infty} (xy)^{-\frac{1}{2}} y^{-\nu-2m-\frac{1}{2}} \\ & G_{33}^{22} \left(y^2 \left/ \begin{matrix} \frac{1}{2}+m, \frac{1}{2}-n, 1-l+\mu/2+\nu/2+m+n \\ \frac{1}{2}+\mu/2+\nu/2+m+n \pm n, k \end{matrix} \right. \right) dy \end{aligned}$$

On writing $y^2 = u$, $2ydy = du$, and evaluating the integral by a well known integral (1, p. 418)

$$\int_0^{\infty} x^{\rho-1} G_{pq}^{mn} \left(ax \left/ \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx = \frac{\prod_{j=1}^m \Gamma(b_j+\rho) \prod_{j=1}^n \Gamma(1-a_j-\rho)}{\prod_{j=m+1}^q \Gamma(1-b_j-\rho) \prod_{j=n+1}^p \Gamma(a_j+\rho)} a^{\rho}$$

$$\text{where } p+q < 2(m+n) \quad |\arg a| < (m+n-p/2-q/2)\pi,$$

$$-\min_{1 \leq J \leq m} R b_J < R(\rho) < 1 - \max_{1 \leq J \leq n} R a_J,$$

we get

$$\int_0^\infty P(xy) dy \int_0^\infty k(x) f(xy) dx$$

$$= 2^{\mu/2+\nu/2+m-1} x^{-\frac{1}{2}} \frac{\Gamma_x(\frac{1}{2}+\nu/2) \Gamma(\frac{1}{2}+\mu/2) \Gamma(\frac{1}{2}+\nu/2+m+n) \Gamma(\frac{1}{2}+\mu/2+2n)}{\Gamma(1+m+\nu/2-k) \Gamma(1+\mu/2+n-l)}$$

This is a $R_{\lambda}(p,q)$ function

$$\text{Hence, } g(x) = \int_0^\infty P(xy) dy \int_0^\infty k(x) f(xy) dx \text{ is } R_{\lambda}(p,q),$$

3. Roop Narain has given the theorem :

If $f(x)$ is $R_{\nu}(k,m)$ and $P(x)$ satisfies the relation

$$x P(x) = P(1/x) \tag{3.1}$$

$$\text{then } g(x) = 1/x \int_0^\infty f(y) P(y/x) dy \tag{3.2}$$

is also $R_{\nu}(k,m)$.

when $P(x) = x^{-\frac{1}{2}} F(x)$, then from (3.1), we have

$$F(x) = F(1/x) \tag{3.3}$$

and then the theorem is :

$$\text{If } f(x) \text{ is } R_{\nu}(k,m) \text{ and } F(x) \text{ satisfies} \tag{3.3}$$

then the function

$$g(x) = \int_0^\infty \frac{F(\omega)}{\sqrt{\frac{\omega}{x}}} f(x\omega) d\omega. \tag{3.4}$$

is $R_{\nu}(k,m)$.

Here we are giving a new formula for self-reciprocal functions. The interest lies mainly in the result and not in a rigorous proof thereof. Hence only the formal procedure is given here.

$$\text{Let } F(\omega) = \int_0^1 \omega^{s-\frac{1}{2}} G(s) ds$$

where $G(s) = G(1-s)$,

$$\text{then } F(1/\omega) = \int_0^1 \omega^{\frac{1}{2}-s} G(s) ds$$

Now, writing $1-s$ in place of s , we get

$$\begin{aligned} F(1/\omega) &= \int_0^1 \omega^{s-\frac{1}{2}} G(1-s) ds, \\ &= \int_0^1 \omega^{s-\frac{1}{2}} G(s) ds, \\ &= F(\omega). \end{aligned}$$

So that $F(\omega)$ satisfies the relation $F(\omega) = F(1/\omega)$.

$$\begin{aligned} \text{Thus } g(x) &= \int_0^\infty \frac{F(\omega)}{\sqrt{\omega}} f(x\omega) d\omega, \\ &= \int_0^\infty \frac{f(x\omega)}{\sqrt{\omega}} d\omega \int_0^1 \omega^{s-\frac{1}{2}} G(s) ds, \\ &= \int_0^1 G(s) ds \int_0^\infty \omega^{s-1} f(x\omega) d\omega \end{aligned}$$

provided the change in the order of integration is justified.

$$\text{Hence, } g(x) = \int_0^1 x^{-s} G(s) ds \int_0^\infty y^{s-1} f(y) dy.$$

Now, by Mellin's Inversion formula [2], we have

$$\int_0^\infty y^{s-1} f(y) dy = \frac{2^{s/2} \Gamma_x(v/2+s/2+\frac{1}{4}+m\pm m)}{\Gamma(v/2+s/2+\frac{3}{4}+m-k)} \omega(s)$$

Therefore

$$g(x) = \int_0^1 x^{-s} G(s) ds \cdot \frac{2^{s/2} \Gamma_x(v/2+s/2+\frac{1}{4}+m\pm m)}{\Gamma(v/2+s/2+\frac{3}{4}+m-k)} \omega(s)$$

where $\omega(s)$ satisfies (1.5).

Hence,

$$g(x) = \int_0^1 \frac{2^{s/2} \Gamma(v/2 + s/2 + \frac{1}{2} + m \pm m)}{\Gamma(v/2 + s/2 + \frac{3}{2} + m - k)} \phi(s) x^{-s} ds. \quad (3.5)$$

is $R_v(k, m)$ where $\phi(s)$ satisfies (1.5).

This formula is similar to (1.4) except for a constant multiple and the limits of integration.

4. Let us test whether the function given by integral (3.5) is self-reciprocal or not. For simplicity we take the case of generalised Hankel Transform, when $k + m = \frac{1}{2}$,

$$g(x) = \int_0^1 2^{s/2} \Gamma(v/2 + s/2 + \frac{1}{2}) \phi(s) x^{-s} ds,$$

The Hankel transform of $g(x)$ is

$$\begin{aligned} &= \int_0^\infty \sqrt{xy} J_\nu(xy) g(y) dy, \\ &= \int_0^\infty \sqrt{xy} J_\nu(xy) dy \int_0^1 2^{s/2} \Gamma(v/2 + s/2 + \frac{1}{2}) \phi(s) y^{-s} ds, \\ &= \int_0^1 2^{s/2} \Gamma(v/2 + s/2 + \frac{1}{2}) \phi(s) ds \int_0^\infty \sqrt{xy} J_\nu(xy) y^{-s} dy, \end{aligned}$$

provided that the inversion of the order of integration is justified.

Evaluating the integral by a well-known formula (Watson) we get the Hankel transform of $g(x)$

$$= \int_0^1 2^{\frac{1}{2}-s/2} \Gamma(\frac{3}{2} + v/2 - s/2) \phi(s) x^{s-1} ds,$$

On writing s for $1-s$, we get

$$\begin{aligned} g(x) &= \int_0^1 2^{s/2} \Gamma(v/2 + s/2 + \frac{1}{2}) \phi(s) x^{-s} ds, \\ &= g(x) \end{aligned}$$

which shows that $g(x)$ is R_v .

5. If $f(x)$ is $-R_v(k, m)$ the function $g(x)$ given by (3.5) is also $-R_v(k, m)$

But if $f(x)$ is $-R_v(k, m)$ and $\phi(s)$ satisfies the equation $-\phi(s) = \phi(1-s)$

instead of (1.5), then again $g(x)$ is $R_v(k, m)$.

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A GENERALISED HANKEL TRANSFORM

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ABSTRACT

A generalisation of the classical Laplace transform has been very recently, given by Mainra (4) in the form

$$\phi(s) = s \int_0^{\infty} (st)^{-\lambda - \frac{1}{2}} e^{-\frac{1}{2}st} W_{k+\frac{1}{2}, m}(st) f(t) dt.$$

In this paper, we have obtained a theorem, which is the generalisation of the well-known Tricomi's Theorem (6) based on the transform given by Mainra (4), various particular cases of the main Theorem have been given. The enunciation of the theorem runs as follows:

Theorem 1: If $\phi(s) = W[f(t); k + \frac{1}{2}, \lambda, m]$

$$\text{then } s^{1-\lambda} \phi(1/s) = W \left[t^{\nu} \int_0^{\infty} F(y) f(y) dy; k_1 + \frac{1}{2}, \lambda_1, m_1 \right]$$

$$\text{where } F(x) = \left[\sum_{m, -m} \frac{\Gamma(-2m)}{\Gamma(-m-k)} \sum_{r=0}^{\infty} \frac{(m-k)_r}{(1+2m)_r} \frac{\Gamma(m+\nu-\lambda-k+r-\lambda_1-k_1+1)}{\Gamma x (m+\nu-\lambda+r-\lambda_1+1 \pm m_1)} \right.$$

$$\left. \frac{x^{-\lambda+m+r}}{r!} {}_1F_2 \left\{ \begin{matrix} m+\nu-\lambda+r-k_1-\lambda_1+1 \\ m+\nu-\lambda+r-\lambda_1+1 \pm m_1 \end{matrix} ; -x \right\} \right]$$

provided that $R(\nu+1-\lambda-\lambda, \pm m_1 \pm m) > 0$, $R(s) > 0$ and $2m$ is not an integer and the integrals involved are absolutely and uniformly convergent.

1. A generalisation of the classical Laplace transform

$$\phi(s) = S \int_0^{\infty} e^{-st} f(t) dt, R(s) > 0, \quad (1.1)$$

has been recently given by Mainra, V. P. [4], in the form

$$\phi(s) = S \int_0^{\infty} e^{-\frac{1}{2}st} (st)^{-\lambda - \frac{1}{2}} W_{k+\frac{1}{2}, m}(st) f(t) dt, R(s) > 0, \quad (1.2)$$

The relation (1.2) reduces to the Meijer-transform [5]

$$\phi(s) = S \int_0^{\infty} e^{-\frac{1}{2}st} (s)^{-k-\frac{1}{2}} W_{k+\frac{1}{2}, m}(st) f(t) dt, R(s) > 0, \quad (1.3)$$

by putting $\lambda = k$, and (1.2) reduces to the generalisation due to Verma [7] if $\lambda = -m$. Further if $\lambda = k = \pm m$, the relation (1.2) reduces to original Laplace transform (1.1).

We may denote relation (1.2) symbolically as

$$\phi(s; k + \frac{1}{2}, \lambda, m) = W[f(t); k + \frac{1}{2}, \lambda, m]$$

while (1.1) is usually denoted by

$$\phi(s) = f(t).$$

As in the case of Laplace transform, we call $\phi(s)$ the image of $f(t)$ and $f(t)$ the original of $\phi(s)$.

In this paper, we have obtained a theorem, which is the generalisation of the well-known Tricomi's Theorem [6], based on the transform given by Varma [7]. Various particular cases of this theorem have been obtained. This generalised Tricomi's Theorem enables us to obtain an interesting theorem, which is a generalisation of a result of Hankel transform. We are giving the images of some functions, which are useful in the following evaluations.

2. By making use of known integrals involving Whittaker function $W_{k,m}(x)$, we have obtained here the images of certain functions.

(a) Using the integral [2]

$$\int_0^{\infty} x^{l-1} e^{-(\alpha^2 + \frac{1}{2})x} W_{k,m}(x) dx = \frac{\Gamma x(l + \frac{1}{2} \pm m)}{\Gamma(l - k + 1)} {}_2F_1 \left[\begin{matrix} l + \frac{1}{2} \pm m \\ l - k + 1 \end{matrix}; -\alpha^2 \right]$$

provided that $R(l + \frac{1}{2} \pm m) > 0$, $R(\alpha^2 + 1) > 0$, Mainra [4] has obtained that

$$W \left[t^v; k + \frac{1}{2}, \lambda, m \right] = \frac{\Gamma x(v - \lambda + 1 \pm m)}{\Gamma(v - k - \lambda + 1)} s^{-v}, \quad (2.1)$$

provided that $R(v + 1 - \lambda \pm m) > 0$, and $R(s) > 0$.

3. Here we have obtained the originals of some functions, which could be expanded in a series of descending powers of S , by interpreting term by term as in the case of Laplace transform.

$$(a) \text{ If } \phi(s) = \left(\frac{a}{s} \right)^v e^{-\alpha/s} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \left(\frac{a}{s} \right)^{v+r}, \text{ in} \quad (1.2)$$

then interpreting the right hand side term by (2.1),

we get

$$\begin{aligned} W \left[\frac{\Gamma(v - k - \lambda + 1)}{\Gamma x(v - \lambda + 1 \pm m)} (\alpha t)^v {}_1F_2 \left\{ \begin{matrix} v - k - \lambda + 1 \\ v - \lambda + 1 \pm m \end{matrix}; -\alpha t \right\}; k + \frac{1}{2}, \lambda, m \right] \\ = (a/s)^v e^{\alpha/s} \end{aligned} \quad (3.1)$$

provided that $\Re (v-\lambda+1\pm m) > 0$ and $\Re(s) > 0$.

(b) If in (1.2) $\phi(s) = s^{-\rho} e^{\alpha/2s} M_{\nu, \mu}(\beta/s)$,

then since

$$s^{-\rho} e^{\alpha/2s} M_{\nu, \rho}(\beta/s) = \left(\frac{\beta-\alpha}{2}\right)^{\mu+\rho+\frac{1}{2}} \sum_{r=0}^{\infty} \frac{(\frac{1}{2}+\mu-\nu)_r}{(1+2\mu)_r} \left(\frac{2\beta}{\beta-\alpha}\right)^r \frac{1}{r!} \left(\frac{\beta-\alpha}{s}\right)^{\mu+\rho+\frac{1}{2}+r} e^{\frac{-\beta-\alpha}{2s}}$$

We get* on interpreting term by term with the help of (3.1)

$$\begin{aligned} W \left[(\beta t)^{\mu+\frac{1}{2}} t^{\rho} \sum_{r=0}^{\infty} \frac{(\frac{1}{2}+\mu-\nu)_r}{(1+2\mu)_r} \frac{\Gamma(\mu+\rho+3/2-k-\lambda+r)}{\Gamma x(\mu+\rho+3/2-\lambda\pm m+r)} \frac{(\beta t)^r}{r!} \times \right. \\ \left. {}_1F_2 \left\{ \begin{matrix} \mu+\rho+3/2-k-\lambda\pm r \\ \mu+\rho+3/2-\lambda+r\pm m \end{matrix} ; \frac{\alpha-\beta}{2} t \right\} ; k\pm\frac{1}{2}, \lambda, m \right] \\ = s^{-\rho} e^{\alpha/2s} M_{\nu, \mu}(\beta/s); \end{aligned} \quad (3.2)$$

provided that $\Re(\mu+\rho+3/2-\lambda\pm m) > 0$, $\Re(s) > 0$ and $2m$ is not a negative integer.

(c) If in (1.2) $\phi(s) = s^{-\rho} e^{\alpha/2s} W_{\nu, \mu}(\beta/s)$

then since (1, p. 430)

$$s^{-\rho} e^{\alpha/2s} W_{\nu, \mu}(\beta/s) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\nu)} s^{-\rho} e^{\alpha/2s} M_{\nu, \mu}(\beta/s) + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\nu)} s^{-\rho} e^{\alpha/2s} M_{\nu, -\mu}(\beta/s).$$

We get on interpreting the right hand side by (3.2),

$$W \left[\frac{\Gamma(-2\mu)(\beta t)^{\mu+\frac{1}{2}}}{\Gamma(\frac{1}{2}-\mu-\nu)} t^{\rho} \sum_{r=0}^{\infty} \frac{(\frac{1}{2}+\mu-\nu)_r}{(1+2\mu)_r} \frac{\Gamma(\mu+\rho+3/2-\lambda-k+r)}{\Gamma x(\mu+\rho+3/2-\lambda+r\pm m)} \frac{(\beta t)^r}{r!} \times \right.$$

Note:— $\Gamma(a)_r = \frac{\Gamma(a+r)}{\Gamma(a)}$

* The uniform convergence of the series follows from the inequality:—

$${}_1F_2 \left[\begin{matrix} a+r \\ b+r, c+r \end{matrix} ; x \right] \leq \frac{(b)_r (c)_r}{(a)_r} {}_1F_2 \left[\begin{matrix} a \\ b, c \end{matrix} ; x \right] \quad (A)$$

${}_1F_2 \left\{ \begin{matrix} \mu+3/2+\rho-\lambda-k+t \\ \mu+\rho+3/2+\nu-\lambda\pm m \end{matrix} ; \alpha-\beta/2t \right\} +$ (a similar expression with $-\mu$ in place of $+\mu$); $k+\frac{1}{2}, \lambda, m$],

$$s^{-\rho} e^{\alpha/2s} W_{\nu, \mu}(\beta/s); \quad (3.3)$$

provided that $R(\mu+\rho+3/2-\lambda\pm m) > 0$, $R(s) > 0$ and 2μ is not an integer.

In case $\alpha = \beta$, (3.3) reduces to

$$W \left[\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\nu)} \frac{\Gamma(\mu+\rho+3/2-\lambda-k)}{\Gamma(\mu+\rho+3/2-\lambda\pm m)} {}_2F_3 \left\{ \begin{matrix} 1/2+\mu-\nu, \mu+\rho+3/2-\lambda-k \\ 1+2\mu, \mu+\rho+3/2-\lambda\pm m \end{matrix} ; \beta t \right\} \right.$$

$+ (a \text{ similar expression with } -\mu \text{ in place of } +\mu); k+\frac{1}{2}, \lambda, m]$

$$= s^{-\rho} e^{\beta/2s} W_{\nu, \mu}(\beta/s);$$

provided that $R(\mu+\rho+3/2-\lambda\pm m) > 0$, $R(s) > 0$ and 2μ is not an integer.

4. Theorem 1.

$$\text{If } \phi(s) = W \left[f(t); k+\frac{1}{2}, \lambda, m \right],$$

$$\text{then } s^{1-\nu} \phi(1/s) = W \left[\int_0^\infty t^\nu F(ty) f(y) dy; k_1+\frac{1}{2}, \lambda_1, m_1 \right]; \quad (4.1)$$

$$\text{where } F(x) = \left[\frac{\Gamma(-2m)}{\Gamma(-m-k)} \sum_{r=0}^\infty \frac{(m-k)_r}{(1+2m)_r} \frac{\Gamma(m+\nu-\lambda-k+r-\lambda_1-k_1+1)}{\Gamma(\nu+m-\lambda+r-\lambda_1+1\pm m_1)} \frac{x^{-\lambda+m+r}}{r!} \right.$$

$$\left. {}_1F_2 \left\{ \begin{matrix} m+\nu-\lambda+r-k_1-\lambda_1+1 \\ m+\nu-\lambda+r-\lambda_1+1\pm m_1 \end{matrix} ; -x \right\} + \frac{\Gamma(2m)}{\Gamma(m-k)} \sum_{r=0}^\infty \frac{(-m-k)_r}{(1-2m)_r} \right.$$

$$\left. \frac{\Gamma(\nu-m-\lambda+\gamma-k_1-\lambda_1+1)}{\Gamma(\nu-m-\lambda+\gamma-\lambda_1+1\pm m_1)} \right]$$

$$\frac{x^{-\lambda-m+r}}{r!} {}_1F_2 \left\{ \begin{matrix} \nu-m-\lambda+r-k_1-\lambda_1+1 \\ \nu-m-\lambda+r-\lambda_1+1\pm m_1 \end{matrix} ; -x \right\};$$

provided that $R(\nu+1-\lambda-\lambda_1\pm m\pm m_1) \geq 0$, $R(s) > 0$ and $2m$ is not an integer and the integrals involved are absolutely and uniformly convergent.

Proof:—We have

$$\phi(s) = s \int_0^\infty (sy)^{-\lambda-\frac{1}{2}} e^{-\frac{1}{2}sy} W_{k+\frac{1}{2}, m}(sy) f(y) dy.$$

Replacing s by $1/s$ and multiplying by $s^{1-\nu}$, we get

$$s^{1-\nu} \phi(1/s) = \int_0^\infty y^{-\lambda-\frac{1}{2}} s^{-(\nu-\lambda-\frac{1}{2})} e^{-\frac{1}{2}y/s} W_{k+\frac{1}{2},m}(y/s) f(y) dy.$$

Now, INTERPRETING the integral by (3.3), with $-\alpha=\beta=y$, $\nu=k+\frac{1}{2}$, $\mu=m$ and $\rho=\nu-\lambda-\frac{1}{2}$, we get

$$s^{1-\nu} \phi(1/s) = W \left[\int_0^\infty t^\nu \left(\frac{\Gamma(-2m)}{\Gamma(-m-k)} \sum_{r=0}^\infty \frac{(yt)^{-\lambda+m+r} (m-k)_r}{r! (1+2m)_r} \frac{\Gamma(m+\nu-\lambda+\gamma-\lambda_1-k_1+1)}{\Gamma x(m+\nu-\lambda+\gamma-\lambda_1+1 \pm m_1)} \right. \right. \\ \left. {}_1F_2 \left\{ \begin{matrix} m+\nu-\lambda+r-\lambda_1+1-k_1 \\ m+\nu-\lambda+r-\lambda_1+1 \pm m_1 \end{matrix} ; -ty \right\} + \frac{\Gamma(2m)}{\Gamma(m-k)} \sum_{r=0}^\infty \frac{(-m-k)_r (yt)^{-\lambda-m+r}}{(1-2m)_r r!} \right. \\ \left. \frac{\Gamma(\nu-m-\lambda+r-k_1-\lambda_1+1)}{\Gamma x(\nu-m-\lambda+\gamma-\lambda_1+1 \pm m_1)} {}_1F_2 \left\{ \begin{matrix} \nu-m-\lambda+\gamma-k_1-\lambda_1+1 \\ \nu-m-\lambda+\gamma-\lambda_1+1 \pm m_1 \end{matrix} ; -ty \right\} \right) ; k+\frac{1}{2}, \lambda, m \right],$$

provided that $R(\nu+1-\lambda-\lambda_1 \pm m \pm m_1) > 0$, $R(s) > 0$, $2m$ is not an integer and the integrals involved are absolutely and uniformly convergent.

It is well-known that if the series and the integrals are absolutely and uniformly convergent, then the term-by-term interpretation is justified. As regards the absolute and uniform convergence of the series, it can easily be seen with the help of the inequality A of (3) and the absolute and uniform convergence of the integrals has been assumed. Further we may note that $F(y)$ is a continuous function of t and y for $t > 0$, $y > 0$ and its behaviour is similar to that of $\phi_\nu; k, \lambda, m(x)$ which will be discussed later on.

5. Particular cases:—

(i) Let $k_1=k$, $m_1=m$ and $\lambda_1=\lambda$, then we have:

$$\text{If } \phi(s) = W[f(t); k+\frac{1}{2}, \lambda, m]$$

$$\text{then } s^{1-\nu} \phi(1/s) = W \left[\int_0^\infty t^\nu \phi_{\nu; k, \lambda, m}(ty) f(y) dy; k+\frac{1}{2}, \lambda, m \right], \quad (5.1)$$

$$\text{where } \phi_{\nu; k+\frac{1}{2}, \lambda, m}(x) = \frac{\Gamma(-2m)}{\Gamma(-m-k)} \sum_{r=0}^\infty \frac{(m-k)_r}{(1+2m)_r} \frac{\Gamma(m+\nu-2\lambda+\gamma-k+1)}{\Gamma x(m+\nu-2\lambda+\gamma+1 \pm m_1)} \times \\ \frac{x^{-\nu+m+r}}{r!} {}_1F_2 \left\{ \begin{matrix} m+\nu-2\lambda+\gamma-k+1 \\ m+\nu-2\lambda+\gamma+1 \pm m \end{matrix} ; -x \right\} + \frac{\Gamma(2m)}{\Gamma(m-k)} \sum_{r=0}^\infty \frac{(-m-k)_r}{(1-2m)_r} \times \\ \frac{\Gamma(\nu-m-2\lambda+\gamma+1-k)}{\Gamma x(\nu-m-2\lambda+\gamma+1 \pm m)} \frac{x^{-\lambda-m+r}}{r!} {}_1F_2 \left\{ \begin{matrix} \nu-m-2\lambda+\gamma+1-k \\ \nu-m-2\lambda+\gamma+1 \pm m \end{matrix} ; -x \right\}, \quad (5.2)$$

provided that $R(\nu - 2\lambda + 1 \pm 2m) > 0$, $R(s) > 0$, $2m$ is not an integer and the integrals involved are absolutely and uniformly convergent.

$\phi_{\nu; k + \frac{1}{2}, \lambda, m}(x)$ can be expressed in another way, as

$$\phi_{\nu; k + \frac{1}{2}, \lambda, m}(x) = \left[\frac{\Gamma(2m)}{\Gamma(m-k)} \frac{\Gamma(\nu - m - 2\lambda - k + 1)}{\Gamma x (\nu - m - 2\lambda + 1 \pm m)} x^{-\lambda - m} \times \right. \\ \left. {}_2F_3 \left\{ \begin{matrix} 1+k-m, \nu-m-2\lambda-k+1 \\ 1-2m, \nu-m-2\lambda+1 \pm m \end{matrix} ; -x \right\} + (\text{a similar expression with } -m \text{ in} \right. \\ \left. \text{place of } +m) \right]:$$

(ii) On putting $\lambda_1 = k_1 = \pm m_1$, in (4.1), we have :

If $\phi(s) = W[f(t); k, \lambda, m]$

$$\text{then } s^{1-\nu} \phi(1/s) = t^\nu \int_0^\infty \left[\frac{\Gamma(-2m)}{\Gamma(-m-k)} \sum_{r=0}^\infty \frac{(m-k)_r}{(1+2m)_r} \frac{(ty)^{\frac{-\nu-\lambda+m+\gamma}{2}}}{r!} \times \right. \\ \left. J_{\nu-\lambda+m+r}(\sqrt{2}ty) + \frac{\Gamma(2m)}{\Gamma(m-k)} \sum_{r=0}^\infty \frac{(-m-k)_r}{(1-2m)_r} \frac{(ty)^{\frac{-\nu-\lambda-m+r}{2}}}{r!} J_{\nu-\lambda-m+r}(2\sqrt{ty}) \right] \\ f(y) dy;$$

provided that $R(\nu + 1 - \lambda \pm m) > 0$, $R(s) > 0$, $2m$ is not an integer and the integrals involved are absolutely and uniformly convergent.

(iii) Let $\lambda = k = \pm m$, and λ_1, k_1, m_1 be replaced by λ, k, m respectively, then, the first series in $F(x)$ vanishes and second series also vanishes except the first term, and we get

$$\text{If } \phi(s) = f(t)$$

then

$$s^{1-\nu} \phi(1/s) = W \left[\frac{\Gamma(\nu - k - \lambda + 1)}{\Gamma x (\nu + 1 - \lambda \pm m)} t^\nu \int_0^\infty {}_1F_2 \left\{ \begin{matrix} \nu - k - \lambda + 1 \\ \nu + 1 - \lambda \pm m \end{matrix} ; -ty \right\} f(y) dy; \right. \\ \left. k + \frac{1}{2}, \lambda, m \right]$$

provided that $R(\nu + 1 - \lambda \pm m) > 0$, $R(s) > 0$, and the integrals involved are absolutely and uniformly convergent.

(iv) Let $\lambda = k = \pm m$ and $\lambda_1 = k_1 = \pm m_1$, is theorem 1, then we get a well-known *Fricomi's Theorem*.

$$\text{If } \phi(s) = f(t),$$

$$\text{then } s^{1-\nu} \phi(1/s) = \int_0^\infty (ty)^{\nu/2} J_\nu(2\sqrt{ty}) f(y) dy,$$

provided that $R(\nu + 1) > 0$, $R(s) > 0$, and the integrals involved are absolutely and uniformly convergent.

6. A new transform:

Consider the integral equation

$$-g(t) = \sqrt{2} \int_0^{\infty} (ty/2)^{\nu + \frac{1}{2}} \phi_{\nu; k, \lambda, m}(t^2 y^2/4) f(y) dy, \quad (6.1)$$

where $\phi_{\nu; k, \lambda, m}(x)$ is given by (5.2), we shall call this equation 'The generalised Hankel transform' and $g(t)$ the $\phi_{\nu; k, \lambda, m}$ -transform of $f(y)$. On putting $\lambda = k = -m$, this transform reduces to Hankel transform. If we have the equation (6.1) together with

$$f(t) = \sqrt{2} \int_0^{\infty} (ty/2)^{\nu + \frac{1}{2}} \phi_{\nu; k + \frac{1}{2}, \lambda, m}(t^2 y^2/4) g(y) dy,$$

functions $f(t)$ and $g(t)$ will be called a pair of $\phi_{\nu; k, \lambda, m}$ -transforms of each other. Further, if $f(t) = g(t)$, then $f(t)$ will be called self-reciprocal in $\phi_{\nu; k + \frac{1}{2}, \lambda, m}$ -transform.

Theorem 2.

$$\text{If } \phi(s) = W[f(t); k + \frac{1}{2}, \lambda, m], \quad (7.1)$$

$$\text{and } \psi(s) = s^{1-\nu} \phi(1/s) = W[g(t); k + \frac{1}{2}, \lambda, m], \quad (7.2)$$

then $s^{\frac{1}{2}-\nu} f(t^2/2)$ and $t^{\frac{1}{2}-\nu} g(t^2/2)$ are $\phi_{\nu; k, \lambda, m}$ -transforms of each other, provided that $R(\nu+1-2\lambda \pm 2m) > 0$, $R(s) > 0$, $(2m)$ is not an integer and the integrals involved converge absolutely and uniformly.

Proof:—Using (5.1) in the relation (6.1), we get

$$s^{1-\nu} \phi(1/s) = W \left[\int_0^{\infty} t^{\nu} \phi_{\nu; k + \frac{1}{2}, \lambda, m}(ty) f(y) dy; k + \frac{1}{2}, \lambda, m \right],$$

Therefore, by virtue of (7.2), we have

$$g(t) = \int_0^{\infty} t^{\nu} \phi_{\nu; k + \frac{1}{2}, \lambda, m}(ty) f(y) dy. \quad (7.3)$$

Further using (5.1) in the relation (7.2), we obtain

$$s^{1-\nu} \psi(1/s) = W \left[\int_0^{\infty} t^{\nu} \phi_{\nu; k, \lambda, m}(ty) g(y) dy; k + \frac{1}{2}, \lambda, m \right].$$

$$\text{But } s^{1-\nu} \psi(1/s) = \phi(s) = W[f(t); k + \frac{1}{2}, \lambda, m],$$

$$\text{giving } f(t) = \int_0^{\infty} t^{\nu} \phi_{\nu; k + \frac{1}{2}, \lambda, m}(ty) g(y) dy, \quad (7.4)$$

Replacing t by $t^2/2$ and y by $y^2/2$ in equation (7.3) and (7.4) and multiplying each equation by $t^{\frac{1}{2}-\nu}$, we arrive at the result.

8. Corollary:—When $\lambda=k=-m$, $\phi_{\nu; k+\frac{1}{2}, \lambda, m}(x') = x^{-\nu/2} J_{\nu}(2\sqrt{g})$,

and theorem 2 reduces to a known result (3, p. 117),

$$\text{If } \phi(s) = \dot{f}(t)$$

$$\text{and } s^{1-\nu} \phi(1/s) = \dot{g}(t)$$

then $t^{\frac{1}{2}-\nu} \int (t^2/2)$ and $t^{\frac{1}{2}-\nu} g(t^2/2)$ form a pair of Hankel transforms of order ν , provided that $\text{Re}(\nu+1) > 0$, and the integrals involved converge absolutely and uniformly.

9. The behaviour of $\phi_{\nu; k+\frac{1}{2}, \lambda, m}(x)$.

It is* easy to see that

$$\phi_{\nu; k+\frac{1}{2}, \lambda, m}(x) = O(x^{-\lambda \pm m}) \text{ for small } x;$$

$$\text{and } \phi_{\nu; k+\frac{1}{2}, \lambda, m}(x) = O(x^{-\mu_1})$$

where $\mu_1 = \frac{1}{2}(\nu - m - \lambda + \frac{1}{2})$, $1+k+\lambda$ or $\nu - k - \lambda + 1$ for large x .

This follows from the behaviour of ${}_2F_3$.

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$$* {}_2F_3 \left[\begin{matrix} \alpha, \beta \\ \gamma, \delta, \varepsilon \end{matrix} ; -x \right]$$

$$\sim C_1 x^{\frac{1}{2}(\alpha+\beta-\gamma-\delta-\varepsilon+\frac{1}{2})} + C_2 x^{-\alpha} + C_3 x^{-\beta}.$$

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RELATION BETWEEN THE HANKEL TRANSFORM AND THE GENERALISED LAPLACE TRANSFORM

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ABSTRACT

In this paper, we have found relation between Hankel transform and the generalised Laplace transforms introduced by Mainra (4). Various particular cases of the main result have been obtained and the result is illustrated by means of examples.

The Laplace transform of a function $f(t)$ belonging to $L(0, \infty)$ is

$$L[f; p] = \int_0^{\infty} e^{-pt} f(t) dt, \quad [R(p) > 0;] \quad (1.1)$$

while its Hankel transform of order ν is defined by the equation

$$H_{\nu}[f; \xi] = \int_0^{\infty} t f(t) J_{\nu}(\xi t) dt, \quad [\xi > 0;] \quad (1.2)$$

Tricomi [9] discovered a relation between Hankel and the Laplace transforms. Recently Bhonsle [1] has established a relation between the Laplace transform of $t^{\mu} f(t)$ and the Hankel transform of $f(t)$. Verma [12] has obtained relations between Hankel transform and the generalised Laplace transforms, one given by Meijer (1940) and the other given by Varma (1951). The present author [8] has also established relations between Hankel transform and the generalised Laplace transforms, one being Whittaker transform (1947) and the other the one due to Meijer (1941).

The object of this note is to find a relation between the Hankel transform and the generalised Laplace transform given by Mainra [4].

Recently, Mainra has developed a theory of the generalised Laplace transform by introducing a new transform, which is more general in nature and form.

He has defined the transform thus :

$$\phi(p) = p \int_0^{\infty} e^{-\frac{1}{2}pt} W_{k+\frac{1}{2}, m}(pt) (pt)^{-\lambda-\frac{1}{2}} f(t) dt, \quad [R(p) > 0] \quad (1.3)$$

provided of course, the integral on the right exists, we express this relation in the form :

$$V_{k+\frac{1}{2}, m} [f; p] = \int_0^{\infty} e^{-\frac{1}{2} pt} W_{k+\frac{1}{2}, m} (pt) (pt)^{-\lambda-\frac{1}{2}} f(t) dt, [R(p) > 0]. \quad (1.4)$$

When $\lambda = k$, $\lambda = -m$, and $\lambda = k = \pm m$;

(1.3) reduces to the Meijer Transform, generalised Laplace transform and the original Laplace transform (1.1) respectively.

2. *Theorem 1.* If $f(t)$ and $H_v [f; \xi]$ belong to $L(0, \infty)$

and if $R(p) > 0$, $|p| > |\xi|$ and $(\mu + v + 1 - \lambda \pm m) > 0$, then

$$V_{k+\frac{1}{2}, m} [t^\mu f; p] = \int_0^{\infty} \psi(p, \xi) H_v [f; \xi] d\xi, \quad (2.1)$$

$$\text{where } \psi(p, \xi) = \frac{\xi^{v+1} \Gamma(\mu + v + 1 - \lambda \pm m)}{2^v \Gamma(v+1) \Gamma(\mu + v + 1 - \lambda - k)} p^{-\mu-v-1} \times$$

$${}_2F_3 \left[\begin{matrix} \frac{\mu+v+1-\lambda \pm m}{2}, \frac{\mu+v+2-\lambda \pm m}{2} \\ v+1, \frac{\mu+v+1-\lambda-k}{2}, \frac{\mu+v+2-\lambda-k}{2} \end{matrix} ; -\xi^2/p^2 \right]. \quad (2.2)$$

Proof.: Since f belongs to $L(0, \infty)$, we have, by Hankel's Inversion theorem [7],

$$f(t) = \int_0^{\infty} \xi H_v [f; \xi] J_v (\xi t) d\xi. \quad (2.3)$$

Now, multiplying both sides by

$$t^\mu e^{-\frac{1}{2} pt} W_{k+\frac{1}{2}, m} (pt) (pt)^{-\lambda-\frac{1}{2}}$$

and integrating between $(0, \infty)$, after inverting the order of integration, which is permissible due to the absolute convergence of integrals involved under conditions stated in the theorem, we get

$$V_{k+\frac{1}{2}, m} [t^\mu f; p] = \int_0^{\infty} \xi H_v [f; \xi] W_{k+\frac{1}{2}, m} [t^\mu J_v (\xi t); p] d\xi \quad (2.4)$$

Again, $W_{k+\frac{1}{2}, m} [t^\mu J_\nu (\xi t); p]$ is evaluated by expanding $J_\nu (\xi t)$ and using, an integral due to Goldstein [3], viz.

$$\int_0^\infty x^{l-1} e^{-(\alpha^2 + \frac{1}{2})x} W_{k, m}(x) dx = \frac{\Gamma x (l \pm m + \frac{1}{2})}{\Gamma (l - k + 1)} {}_2F_1 \left[\begin{matrix} l \pm m + \frac{1}{2} \\ l - k + 1 \end{matrix}; -\alpha^2 \right],$$

where $(l \pm m + \frac{1}{2}) > 0$, $R(\alpha^2 - 1) > 0$ and $|\alpha| < 1$, we get

$$W_{k+\frac{1}{2}, m} [t^\mu J_\nu (\xi t), p] = \frac{\xi^\nu \Gamma x (\mu + \nu + 1 - \lambda \pm m)}{2^\nu \Gamma (\nu + 1) \Gamma (\mu + \nu + 1 - \lambda - k)} p^{-\mu - \nu - 1} \times {}_4F_3 \left[\begin{matrix} \frac{\mu + \nu - \lambda + 1 \pm m}{2}, \frac{\mu + \nu - \lambda + 2 \pm m}{2} \\ \nu + 1, \frac{\mu + \nu + 1 - \lambda - k}{2}, \frac{\mu + \nu + 2 - \lambda - k}{2} \end{matrix}; \frac{-\xi^2}{p^2} \right],$$

where $R(\mu + \nu - \lambda + 1 \pm m) > 0$, $R(p) > 0$, $|p| > |\xi|$,

Thus (2.4) reduces to

$$V_{k+\frac{1}{2}, m} [t^\mu f; p] = \int_0^\infty \psi(p, \xi) H_\nu [f; \xi] d\xi,$$

where

$$\psi(p, \xi) = \frac{\xi^{\nu+1} \Gamma x (\mu + \nu - \lambda + 1 \pm m)}{2^\nu \Gamma (\nu + 1) \Gamma (\mu + \nu + 1 - \lambda - k)} \times {}_4F_3 \left[\begin{matrix} \frac{\mu + \nu - \lambda + 1 \pm m}{2}, \frac{\mu + \nu - \lambda + 2 \pm m}{2} \\ \nu + 1, \frac{\mu + \nu + 1 - \lambda - k}{2}, \frac{\mu + \nu + 2 - \lambda - k}{2} \end{matrix}; \frac{-\xi^2}{p^2} \right],$$

Hence the theorem.

Particular cases.

Case (i), when $\lambda = -m$, we get a well-known result due to a Verma, viz.

If $f(t)$ and $H_\nu [f, \xi]$ belong to $L(0, \infty)$ and if

$R(\mu + \nu + m + 1 \pm m) > 0$, $R(p)$, $R(\xi) > 0$ then

$$V_{k+\frac{1}{2}, m} [t^\mu f; p] = \int_0^\infty \phi(p, \xi) H_\nu [f, \xi] d\xi,$$

where

$$\phi(p, \xi) = \frac{\xi^{v+1} \Gamma(\mu+v+m+1 \pm m)}{2^v \Gamma(v+1) \Gamma(\mu+v+1+m-k)} p^{-\mu-v-1} \times {}_4F_3 \left[\begin{matrix} \frac{\mu+v+m+1 \pm m}{2}, \frac{\mu+v+m+2 \pm m}{2} \\ v+1, \frac{\mu+v+1+m-k}{2}, \frac{\mu+v+2+m-k}{2} \end{matrix} ; \frac{-\xi^2}{p^2} \right],$$

case (ii) when $\lambda=k$, we get a result due to Singh [8] viz.

If $f(t)$ and $H_v[f; \xi]$ belong to $L(o, \infty)$ and if

$R(\mu+v-k+1 \pm m) > 0$, $R(p)$, $R(\xi) > 0$, then

$$W_{k+\frac{1}{2}, m}[t^\mu f; p] = \int_0^\infty \theta(p; \xi) H_v[f; \xi] d\xi,$$

where

$$\theta(p, \xi) = \frac{\xi^{v+1} \Gamma(\mu+v-k+1 \pm m)}{2^v \Gamma(v+1) \Gamma(\mu+v-2k+1)} p^{-\mu-v-1} \times {}_4F_3 \left[\begin{matrix} \frac{\mu+v-k+1 \pm m}{2}, \frac{\mu+v-k+2 \pm m}{2} \\ v+1, \frac{\mu+v+1-2k}{2}, \frac{\mu+v+2-2k}{2} \end{matrix} ; \frac{-\xi^2}{p^2} \right],$$

Case (iii). When $\lambda=k=\pm m$, we get a well-known result due to Bhonsle, viz.

$$L[t^\mu f; p] = \int_0^\infty k(p, \xi) H_v[f; \xi] d\xi,$$

where

$$k(p, \xi) = \Gamma(\mu+v+1) \xi (p^2 + \xi^2)^{-\frac{1}{2}} (\mu+1) p^{-\nu} \left[\frac{p}{\sqrt{p^2 + \xi^2}} \right],$$

$$R(\mu) > -1, \quad R(\mu+v) > -1; R(p) > 0,$$

For illustrating the theorem, the following examples are worth mention :

Example 1. Let $f(t) = t^{l-1} e^{-\beta t}$, $|\beta| > 0$.

$$\text{then } H_v[f; \xi] = \Gamma(l+v+1) (\beta^2 + \xi^2)^{-\frac{1}{2}} (l+1) p^{-\nu} \left(\frac{\beta}{\sqrt{\beta^2 + \xi^2}} \right),$$

where $R(l+v+\frac{1}{2}) > 0$,

[2, p. 29].

and

$$V_{k+\frac{1}{2}, m} [t^\mu f; p] = \frac{\Gamma x (\mu+l-\lambda \pm m)}{\Gamma (\mu+l-k-\lambda)} \frac{p^{m-\lambda}}{(\beta+p) \mu+l+m-\lambda} \times$$

$${}_2F_1 \left(\begin{matrix} \mu+l+m-\lambda \\ \mu+l-k-\lambda \end{matrix} ; m-k \frac{\beta}{\beta+p} \right),$$

$$R(\rho+2\beta) > 0.$$

(This is due to Mainra).

Hence, by the theorem,

$$V_{k+\frac{1}{2}, m} [t^\mu f; p] = \int_0^\infty \psi(p, \xi) H_\nu [f; \xi] d\xi,$$

we have

$$\int_0^\infty \xi^{\nu+1} (\beta^2 + \xi^2)^{-\frac{1}{2}(l+1)} p^{-\nu} \left(\sqrt{\frac{\beta}{\beta^2 + \xi^2}} \right)$$

$${}_4F_3 \left[\begin{matrix} \frac{\mu+\nu+1-\lambda \pm m}{2}, \frac{\mu+\nu+2-\lambda \pm m}{2} \\ \nu+1, \frac{\mu+\nu-\lambda-k+1}{2}, \frac{\mu+\nu+2-k-\lambda}{2} \end{matrix} ; \frac{-\xi^2}{p^2} \right] d\xi$$

$$= \frac{2^\nu p^{\mu+\nu+1} \Gamma(\nu+1) \Gamma(\mu+\nu+1-\lambda-k) \Gamma x (b+l-\lambda \pm m) p^{m-\lambda}}{\Gamma(\mu+l-k-\lambda) \Gamma x (\mu+\nu+1-\lambda \pm m) \Gamma(l+\nu+1) (\beta+p)^{\mu+l-\lambda+m}}$$

$${}_2F_1 \left[\begin{matrix} \mu+l+m-\lambda, m-k \\ \mu+l-k-\lambda \end{matrix} ; \frac{\beta}{\beta+p} \right].$$

Example 2. Let us take $f(t) = x^{-1} e^{-\alpha/t}$.

Then $H_\nu [f; \xi] = J_\nu \left(\sqrt{2\alpha\xi} \right) k_\nu \left(\sqrt{2\alpha\xi} \right)$.

$$R(\alpha) > 0, \quad [2, p. 30].$$

and

$$V_{k+\frac{1}{2}, m} [t^\mu f; p] = V_{k+\frac{1}{2}, m} [t^{k+\lambda} f; p] \quad (\mu=k+\lambda),$$

$$\text{Hence} \quad = 2\alpha^k p^{1-\lambda} k_{2m} \left(2\sqrt{\alpha p} \right).$$

$$\int_0^\infty \xi^{\nu+1} J_\nu \left(\sqrt{2\alpha\xi} \right) k_\nu \left(\sqrt{2\alpha\xi} \right)$$

$${}_4F_3 \left[\begin{matrix} \frac{\lambda+\nu+1 \pm m}{2}, \frac{\lambda+\nu+2 \pm m}{2} \\ \nu+1, \frac{\lambda+\nu+1-k}{2}, \frac{\lambda+\nu+2-k}{2} \end{matrix} ; \frac{-\xi^2}{p^2} \right] d\xi$$

$$= \frac{2^{\nu+1} \alpha^k p^{1-\lambda k_{2m}} (2\alpha p)^{2\Gamma(\nu+1)} \Gamma(\nu+1)}{\Gamma(\lambda+k+\nu-\lambda+1 \pm m)} p^{k+\lambda+\nu+1},$$

Thus the theorem is verified.

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STUDIES ON THE YIELD OF RICE AS AFFECTED BY SOIL DROUGHT AND POTASSIUM APPLICATION—I.

By

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ABSTRACT

Data pertaining to yield as affected by different treatments have been presented and discussed. The main findings of the experiment are given below :

- (1) Stoppage of irrigation a few days before the harvesting of an early maturing variety, initiates higher production of grain and also of straw.
- (2) Application of potassium in split doses has been found to increase the yield of grain as well as of straw of drought affected plants.
- (3) I_0K_2 (drought and application of potassium in two instalments) and I_0K_3 (drought and application of potassium in three instalments) have been found the best treatment combinations for the production of grain and straw respectively.

INTRODUCTION

Adequate supply of irrigation, particularly at critical stages of growth and development, is most essential for successful cultivation of the rice crop. Irrigation facilities are being increased at a fast rate, but they are still too inadequate to meet the general requirement and the bulk of rice crop in India largely depends upon the vagaries of rainfall. Drought conditions are always prevalent in one or the other part of the country as a result of which rice yield fluctuates from place to place and year to year. No serious attempt appears to have been made in this country to study the relationship between drought conditions and the growth and yield of the rice crop.

Potassium has been found to cause hardness in cereals though this view is still a matter of controversy. Smith and Butler (1921) found negligible effect on the carbohydrate formation in blue stem wheat when raised in a solution deficient in potassium. On the other hand, Eckstein (1939) was of the view that shortage of potassium in the leaf leads to low rates of carbon dioxide assimilation. Sircar and Datta (1957) observed harmful effects on plants growing under potassium deficiency. Hartwell (1927) found that different species responded differently when supplied with the same amount of potassium under similar conditions of growth. Gregory (1937) has discussed the variable role of potassium in the plant. He concluded that the response due to potassium will depend upon the relative concentration of other elements, particularly sodium and calcium in the plant tissues. Literature so far available on the role of potassium in the plants reveals that it induces hardness in tissues and thereby the plants can resist adverse environmental conditions to a greater extent.

The present investigation was undertaken to get further informations regarding the role of potassium and created drought conditions on growth and yield of rice crop.

EXPERIMENTAL METHOD

In order to get reliable informations, the entire experiment was carried out in medium sized cement pots under controlled conditions. Soil, for filling of the pots, was taken from a field where no manurial trial had been conducted for the last 5 years. It was sieved after breaking all clods and removing grasses, etc. and was thoroughly mixed with the help of a spade. 20 kg. of this prepared soil was filled in each pot, and soaked well with a known amount of water.

Seedlings of a short duration variety (N_{22}) were raised under natural conditions of the farm, and three seedlings of uniform size were transplanted in each pot on 3rd July 1961. The plants were grown under drought and normally irrigated conditions and were arranged in a randomised block design, having 3 replications and 8 combination treatments of irrigation and potassium application as main factors, the composition of which is given below :

- (1) I_1K_0 Normal irrigation was provided throughout the life cycle of the plant, but no potassium was applied.
- (2) I_1K_1 Irrigation was provided normally, and the potassium was applied at the rate of 40 lb./ acre in a single dose just after one week of transplantation.
- (3) I_1K_2 Provision was made for normal irrigation, but the amount of potassium was divided in two parts, the first one was applied one week after transplanting and the other one month later.
- (4) I_1K_3 Normal irrigation was continued as previously, and the whole amount of potassium was split in three equal doses and each one was applied at an interval of one month starting from one week after transplanting.
- (5) I_0K_0 Irrigation stopped at the grain maturation stage i. e., 60 days after transplanting and no potassium was applied.
- (6) I_0K_1 Irrigation was stopped as in the above treatment and potassium was applied at the rate of 40 lb./ acre in a single dose one week after the transplanting.
- (7) I_0K_2 Irrigation was stopped in a similar way, but the amount of potassium was divided in two parts, the first was applied a week after transplantation and the other one month after the first application.
- (8) I_0K_3 Here also the irrigation was stopped as in previous cases, but the amount of potassium was applied in three equal instalments, each one was applied at an interval of one month starting from a week after transplanting.

Note : Under normal irrigation, each pot was supplied with equal and measured, but liberal volume of water. While stopping the irrigation in pots under drought treatment, the initial level of water was kept two inches above the soil surface. All the pots, viz., I_0 and I_1 were taken to the glass house in order to protect them from rain and dew of outside and were kept under exactly similar conditions.

Each pot received basal dressings of ammonium sulphate and super phosphate at the rate of 40 lb. and 60 lb. per acre respectively.

Regular observations were recorded on the growth habits and deficiency symptoms of plants and soil. Day to day meteorological observations were also taken for the convenience of discussion. Plants were harvested when they attained maturity. It was found that the normally irrigated plants could be harvested 5 days earlier than the drought affected ones.

EXPERIMENTAL RESULTS

Grain and straw yields due to different treatments are shown in table 1 and their statistical analyses are presented in tables 2 and 3. Analysis of variance (table 2) apparently indicates that the main factors *i. e.* irrigation and potassium along with their interactions influence the rice yields, both of grain and straw, very significantly.

TABLE 1
Effect of Different Treatments on the yield of rice crop in Gram per plant

Treatments	Replication I	Replication II	Replication III	Mean
<i>GRAIN</i>				
1. I_1K_0	15.0	15.1	15.8	15.3
2. I_1K_1	13.0	14.9	13.8	13.9
3. I_1K_2	12.3	14.4	13.2	13.3
4. I_1K_3	13.5	15.6	14.4	14.5
5. I_0K_0	14.6	16.7	15.5	15.6
6. I_0K_1	15.7	17.5	16.6	16.6
7. I_0K_2	19.8	21.6	21.0	20.8
8. I_0K_3	15.6	17.6	16.6	16.6
<i>STRAW</i>				
1. I_1K_0	67.4	65.4	66.4	66.4
2. I_1K_1	63.5	57.7	63.9	61.7
3. I_1K_2	63.6	57.9	64.2	61.9
4. I_1K_3	74.2	68.5	74.8	72.5
5. I_0K_0	87.1	80.4	88.7	85.4
6. I_0K_1	85.2	81.5	86.8	84.5
7. I_0K_2	88.0	82.2	88.4	86.2
8. I_0K_3	88.0	80.1	90.2	85.1

TABLE 2
Analysis of Variance

Due to	D.F.	M.S. for grain	M.S. for straw
Replication	2	6.04	91.38
Irrigation (I)	1	59.53**	2382.04**
Potassium (K)	3	4.09**	44.87**
I × P	3	14.18**	34.13**
Error	14	0.15	1.84
Total	23		

**Significant at 1% level.

TABLE 3
Interaction between Potassium and Irrigation

	K ₀	K ₁	K ₂	K ₃	Mean
<i>GRAIN</i>					
I ₁	15.3	13.9	13.3	14.5	14.3
I ₀	15.6	16.6	20.8	16.6	17.4
Mean	15.5	15.3	17.1	15.6	
<i>STRAW</i>					
I ₁	66.4	61.7	61.9	72.5	65.6
I ₀	85.4	84.5	86.2	86.1	85.6
Mean	75.9	73.1	74.1	79.3	

C.D. for Grain — I = 0.342, K = 0.471, I × K = 0.663.

C.D. for Straw — I = 1.030, K = 1.669, I × K = 2.854.

A study of the experimental results shown in table 3 clearly reveals that stopping of irrigation a few days before the harvesting of rice plants results into higher yields, both of grain and straw ; which means in other words, that even though the irrigation is the most important factor of growth and yield of rice, its efficiency is much more increased when it is discontinued in the later part of the life cycle of the crop. Among different methods of application of potassium, K₂, in which half of the amount was applied after one week of transplanting and the other half one month later, gave maximum yield of grain (significant at 1% level

over other methods of application), but failed to show any marked response in case of straw and it was only K_3 (the whole amount applied in three instalments at an interval of one month starting just after one week of transplanting) which resulted in significantly higher yields of it over other methods of application.

Experimental results on the relationships of potassium application and drought conditions appear to be of greater interest. A created drought (I_0) initiated the production of highest amount of paddy grain when supplied with potassium in two split doses i. e. K_2 . It can also be seen that grain yields were always higher in case of interactions of potassium with created drought than with normally irrigated ones. More or less the same trend was also observed in case of the straw yield. Here, though the critical difference between different combinations of created drought and potassium application (I_0K) was not found significant, but each one of them showed its superiority very markedly over the combinations of normal irrigation and potassium application (I_1K).

DISCUSSION

Experimental findings have clearly shown that the drought affected plants yielded more of grain and straw than normally irrigated ones. This result is of interest and is explained on the basis of the regular observations and findings made in course of the experimentation. It was observed that drought affected plants ripened 5 days later than the normally irrigated ones. It seems that during this prolonged period, they continued their growth under the restricted soil moisture supply and hence it is quite possible for them to yield more. In addition to this, a low soil moisture might have helped in better aeration and thereby extension of more root system in the direction of available moisture and nutrients, thus helping the plant in overcoming the severe drought injury and in completing its life cycle; though a little bit late, but before the level of soil moisture could reach the point of permanent wilting. Further more, in case of normally irrigated plant, presence of enough moisture in the soil might have caused better supply of nitrogen, thus leading to more tillering and succulence and thereby the energy which should have gone in the formation of grain might have been diverted in new growth, though here also the dry matter of the whole plant did not increase. This aspect of better supply of nitrogen is also proved by the fact that normally irrigated crop matured earlier than the drought affected ones which is in close agreement to Russell's statement of 1961.

Regarding the effect of potassium on the yield of paddy crop, it is evident that normally irrigated plants did not respond to its application as those affected by drought condition. It may be that nitrogen content in the plant increased by adequate moisture supply, but the same amount of potassium might have become inadequate, thus causing harmful effect on the plant. Russell (1961) has also stated that plant tissues might be killed if potassium supply did not catch up the nitrogen supply. On the other hand, it is also possible that abundance of water in the soil might have caused accumulation of potassium in the plant tissues; this might have lowered the supply of other micronutrients and thereby the yield was reduced. Walsh and Donohoe (1945) also reported similar results. Split doses of potassium in case of drought affected plants have given best performance, this is quite expected because it might have helped the plant in maintaining the normal or desirable potassium status practically at all stages of the growth.

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SUMMARY

Yield of paddy crop mostly depends upon the adequate and timely supply of irrigation. Drought conditions in one or the other forms are still of frequent occurrence in many parts of India, as a result of which crop like paddy suffers to a great extent. It appears from the existing literature that application of potassium causes some hardiness in the plant, and therefore, in the present investigation an attempt has been made to see the extent to which its application is helpful to the drought affected rice crop in maintaining its growth and yield. The main conclusions drawn from the results are given below :

- (1) Stopping of irrigation a few days before harvesting initiates higher yields both of grain and straw.
- (2) In the case of drought affected plants, split doses of potassium appear to give better response (both of grain and straw) than its application in a single dose.
- (3) As regards the grain yield, a created drought and application of potassium in two instalments (I_0K_2) has been found to be the best treatment. Further, the yield of straw was significantly improved when potassium was applied in three split doses.

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ON THE HARMONIC SUMMABILITY OF THE DERIVED FOURIER SERIES AND ITS CONJUGATE SERIES

By

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ABSTRACT

Prasad and Singh (1952) have obtained a general criterion for the strong Cesàro summability of the derived Fourier series and its conjugate series. Under slightly relaxed condition of their criterion, we have studied in this paper the harmonic summability of these series.

1.1. Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + p_2 + \dots + p_n.$$

The sequence-to-sequence transformation

$$(1.1.1) \quad t_n = \sum_{v=0}^n \frac{p_v s_{n-v}}{P_n}, \quad (P_n \neq 0),$$

defines the sequence $\{t_n\}$ of Nörlund means of the sequence $\{s_n\}$, generated by the sequence of coefficients $\{p_n\}$. The series $\sum a_n$ is said to be summable (N, p_n) to the sum s if $\lim_{n \rightarrow \infty} t_n$ exists and equals s . In the special case in which

$$(1.1.2) \quad p_n = \frac{1}{n+1},$$

and, therefore,

$$(1.1.3) \quad P_n \sim \log n, \text{ as } n \rightarrow \infty,$$

t_n reduces to the familiar 'harmonic mean' of $\{s_n\}$, which is defined independently as follows :

A sequence $\{s_n\}$ is said to be summable by harmonic means, if

$$(1.1.4) \quad \lim_{n \rightarrow \infty} \frac{1}{\log n} \sum_{k=0}^n \frac{s_{n-k}}{k+1},$$

exists and is finite. This method of summability which belongs to the class of generalized regular Nörlund [4] methods, was introduced for the first time by M. Riesz [6], who proved that every series summable by harmonic means, is summable (G, δ) , $\delta > 0$.

1.2. Let $f(t)$ be a continuous function of bounded variation, periodic with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$ and have a derivative $f'(x)$ at $t=x$. Let the Fourier series corresponding to $f(t)$ be

$$(1.2.1.) \quad \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt).$$

The series

$$(1.2.2) \quad \sum_{n=1}^{\infty} n (b_n \cos nt - a_n \sin nt),$$

which is obtained by differentiating (1.2.1) term by term, is called the first derived series or the derived Fourier series of $f(t)$.

The series conjugate to (1.2.2) is

$$(1.2.3) \quad \sum_{n=1}^{\infty} n (a_n \cos nt + b_n \sin nt).$$

We write

$$g(t) = g(t, x) = f(x+t) - f(x-t) - 2tf'(x),$$

$$h(t) = h(t, x) = f(x+t) + f(x-t) - 2f(x),$$

$$\mathcal{N}_n(t) = \frac{1}{2\pi \log n} \sum_{k=0}^n \frac{1}{k+1} \frac{\sin(n-k+\frac{1}{2})t}{\sin \frac{1}{2}t},$$

$$\bar{\mathcal{N}}_n(t) = \frac{1}{2\pi \log n} \sum_{k=0}^n \frac{1}{k+1} \frac{\cos(n-k+\frac{1}{2})t}{\sin \frac{1}{2}t},$$

$$G(t) = \int_0^t |dg(u)|,$$

$$\chi(t) = \int_0^t |dh(u)|,$$

and

$$\begin{aligned} H(x) &= \frac{-1}{4\pi} \int_0^\pi h(t) \operatorname{cosec}^2 \frac{1}{2}t dt \\ &= \lim_{\epsilon \rightarrow 0} \left\{ \frac{-1}{4\pi} \int_\epsilon^\pi h(t) \operatorname{cosec}^2 \frac{1}{2}t dt \right\}. \end{aligned}$$

We denote the integral

$$\frac{-1}{4\pi} \int_{1/n}^\pi h(t) \operatorname{cosec}^2 \frac{1}{2}t dt \text{ by } H_n(x),$$

so that as $n \rightarrow \infty$, $H_n(x) \rightarrow H(x)$, viz.,

$$(1.2.4) \quad \lim_{n \rightarrow \infty} H_n(x) = H(x).$$

If the function $f(x)$ is of bounded variation, then $H(x)$ exists for almost all value of x [Zygmund, 9, p. 146].

2.1. Prasad and Singh [5] have discussed the strong Cesàro summability of the derived series of the Fourier series and its conjugate series under the conditions

$$(2.1.1) \quad \int_0^t |dg(u)| = o \left\{ t/(\log 1/t)^{1+\varepsilon} \right\}, \quad (\varepsilon > 0),$$

and

$$(2.1.2) \quad \int_0^t |dh(u)| = o \left\{ t/(\log 1/t)^{1+\varepsilon} \right\}, \quad (\varepsilon > 0),$$

respectively. The object of this paper is to study the harmonic summability of the corresponding series under the same condition except that ε is taken to be zero in our case.

2.2. In what follows we shall prove the following theorems :

Theorem 1. *The derived series of the Fourier series of the function $f(x)$ is summable by harmonic means to the sum $f'(x)$ at a point x at which*

$$(2.2.1) \quad G(t) = o(t/\log 1/t),$$

where $f'(x)$ denotes the first generalized differential coefficient of $f(t)$ at $t=x$.

Theorem 2. *The conjugate derived series of the Fourier series of the function $f(x)$ is summable by harmonic means to the sum*

$$(2.2.2) \quad -\frac{1}{4\pi} \int_0^\pi h(t) \operatorname{cosec}^2 \frac{1}{2} t dt,$$

at every point x at which this integral exists and

$$(2.2.3) \quad \chi(t) = o(t/\log 1/t), \text{ as } t \rightarrow +0.$$

It is evident that under the condition (2.2.3),

$$(2.2.4) \quad \chi(t) = o(t), \text{ as } t \rightarrow 0.$$

2.3. In order to prove the theorems, we shall require the following Lemmas :

Lemma 1 [1] *If $0 < t < \pi$, then*

$$(2.3.1) \quad \left| \sum_{k=0}^n \cos(k+1)t/(k+1) \right| \leq A(1 + \log 1/t),$$

where A is an absolute constant.

Lemma II. [8, p. 440] For all values of n and t ,

$$(2.3.2) \quad \left| \sum_{k=0}^n \sin (k+1) t / (k+1) \right| \leq \frac{1}{2} \pi + 1.$$

3.1. Proof of the theorem 1.

Denoting by $\sigma_n(x)$ the sum of the first n terms of the series (1.2.2) at the point $t=x$, we get

$$\begin{aligned} \sigma_n(x) &= \frac{1}{2\pi} \int_0^{2\pi} \left\{ \frac{d}{dx} \frac{\sin (n+\frac{1}{2})(x-u)}{\sin \frac{1}{2}(x-u)} \right\} f(u) du \\ &= \frac{-1}{2\pi} \int_0^{2\pi} f(u) \left\{ \frac{d}{du} \frac{\sin (n+\frac{1}{2})(x-u)}{\sin \frac{1}{2}(x-u)} \right\} du \\ (3.1.1) \quad &= \frac{-1}{2\pi} \int_0^\pi \{ f(x+t) - f(x-t) \} \left\{ \frac{d}{dt} \frac{\sin (n+\frac{1}{2})t}{\sin \frac{1}{2}t} \right\} dt. \end{aligned}$$

Now integrating by parts the right-hand side of (3.1.1), we obtain

$$\begin{aligned} \sigma_n(x) &= \frac{1}{2\pi} \int_0^\pi \frac{\sin (n+\frac{1}{2})t}{\sin \frac{1}{2}t} d \{ f(x+t) - f(x-t) \} \\ &= \frac{1}{2\pi} \int_0^\pi \frac{\sin (n+\frac{1}{2})t}{\sin \frac{1}{2}t} \cdot d g(t) + f'(x). \end{aligned}$$

Hence

$$\sigma_n(x) - f'(x) = \frac{1}{2\pi} \int_0^\pi \frac{\sin (n+\frac{1}{2})t}{\sin \frac{1}{2}t} d g(t),$$

therefore

$$\sigma_{n-k}(x) - f'(x) = \frac{1}{2\pi} \int_0^\pi \frac{\sin (n-k+\frac{1}{2})t}{\sin \frac{1}{2}t} d g(t).$$

Making use of the formula (1.1.4), we obtain

$$\begin{aligned} &\frac{1}{\log n} \sum_{k=0}^n \frac{1}{k+1} \left\{ \sigma_{n-k}(x) - f'(x) \right\} \\ &= \frac{1}{\log n} \sum_{k=0}^n \frac{1}{k+1} \cdot \frac{1}{2\pi} \int_0^\pi \frac{\sin (n-k+\frac{1}{2})t}{\sin \frac{1}{2}t} d g(t). \end{aligned}$$

$$\begin{aligned}
&= \int_0^\pi dg(t) \cdot \frac{1}{2\pi \log n} \sum_{k=0}^n \frac{1}{k+1} \frac{\sin(n-k+\frac{1}{2})t}{\sin \frac{1}{2}t} \\
&= \int_0^\pi dg(t) \cdot N_n(t).
\end{aligned}$$

In order to prove the theorem we have to show that

$$(3.1.2) \quad \int_0^\pi dg(t) \cdot N_n(t) = o(1),$$

as $n \rightarrow \infty$, when (2.2.1) holds.

We set

$$\begin{aligned}
\int_0^\pi dg(t) N_n(t) &= \left(\int_0^{1/n} + \int_{1/n}^\delta + \int_\delta^\pi \right) dg(t) N_n(t) \\
&= Z_1 + Z_2 + Z_3,
\end{aligned}$$

say, where δ is small but fixed.

Now

$$\begin{aligned}
Z_1 &= \int_0^{1/n} dg(t) \cdot N_n(t) \\
&= O \left\{ \int_0^{1/n} |dg(t)| \cdot |N_n(t)| \right\} \\
&= O \left\{ \int_0^{1/n} |dg(t)| \cdot \frac{1}{\log n} \sum_{k=0}^n \frac{1}{k+1} \frac{|\sin(n-k+\frac{1}{2})t|}{|\sin \frac{1}{2}t|} \right\} \\
&= O \left\{ \int_0^{1/n} |dg(t)| \cdot \frac{1}{\log n} \sum_{k=0}^n \frac{1}{k+1} \frac{(2n-2k+1) |\sin \frac{1}{2}t|}{|\sin \frac{1}{2}t|} \right\} \\
&= O \left\{ \int_0^{1/n} |dg(t)| \cdot \frac{1}{\log n} \sum_{k=0}^n \frac{1}{k+1} \cdot n \right\}
\end{aligned}$$

$$\begin{aligned}
&= O \left\{ \int_0^{1/n} |dg(t)| \cdot n \right\} \\
&= O \left\{ G(1/n) \cdot n \right\} \\
&= o \left\{ n^{-1} (\log n)^{-1} \cdot n \right\} \\
(3.1.3) \quad &= o(1).
\end{aligned}$$

Also by virtue of Riemann-Lebesgue theorem and regularity of the method of summation, we have

$$\begin{aligned}
Z_3 &= \int_{\delta}^{\pi} dg(t) \cdot N_n(t) \\
&= O \left\{ \int_{\delta}^{\pi} |dg(t)| \cdot |N_n(t)| \right\} \\
&= O \left\{ \frac{1}{\log n} \int_{\delta}^{\pi} |dg(t)| \cdot \sum_{k=0}^n \frac{1}{k+1} \left| \frac{\sin(n-k+\frac{1}{2})t}{\sin \frac{1}{2}t} \right| \right\} \\
(3.1.4) \quad &= o(1).
\end{aligned}$$

Lastly in order to show that $Z_2 = o(1)$, we require a suitable estimate for the kernel $N_n(t)$ in the interval (n^{-1}, δ) .

Since

$$\begin{aligned}
N_n(t) &= \frac{1}{2\pi \log n \sin \frac{1}{2}t} \left\{ \sin(n + \frac{3}{2})t \sum_{k=0}^n \frac{\cos(k+1)t}{k+1} \right. \\
&\quad \left. - \cos(n + \frac{3}{2})t \sum_{k=0}^n \frac{\sin(k+1)t}{k+1} \right\},
\end{aligned}$$

by making use of the Lemmas I and II, we get

$$N_n(t) = O \left[\frac{1}{t \log n} \left\{ 1 + \log 1/t \right\} \right].$$

Hence

$$\begin{aligned}
Z_2 &= \int_{1/n}^{\delta} dg(t) N_n(t) \\
&= O \left\{ \int_{1/n}^{\delta} |dg(t)| \cdot |N_n(t)| \right\}
\end{aligned}$$

$$\begin{aligned}
&= O \left\{ \frac{1}{\log n} \int_{1/n}^{\delta} |dg(t)| \cdot 1/t \right\} + O \left\{ \frac{1}{\log n} \int_{1/n}^{\delta} |dg(t)| \cdot 1/t \cdot \log 1/t \right\} \\
&= O \left[\frac{1}{\log n} G(t) \cdot 1/t \right]_{1/n}^{\delta} + O \left[\frac{1}{\log n} \int_{1/n}^{\delta} G(t) \cdot 1/t^2 dt \right] \\
&+ O \left[\frac{1}{\log n} G(t) \cdot 1/t \cdot \log 1/t \right]_{1/n}^{\delta} + O \left[\frac{1}{\log n} \int_{1/n}^{\delta} G(t) \left\{ 1 + \log 1/t \right\} \frac{dt}{t^2} \right] \\
&= O \left[\frac{1}{\log n} \right] + o \left[\frac{1}{(\log n)^2} \right] + o \left[\frac{1}{\log n} \int_{1/n}^{\delta} \frac{dt}{t \log 1/t} \right] \\
&+ O \left[\frac{1}{\log n} \right] + o \left[\frac{1}{\log n} \right] + o \left[\frac{1}{\log n} \int_{1/n}^{\delta} \left\{ \frac{1}{t \log 1/t} + \frac{1}{t} \right\} dt \right] \\
&= o(1) + o \left[\frac{1}{\log n} \int_{1/n}^{\delta} \frac{dt}{t \log 1/t} \right] + o \left[\frac{1}{\log n} \int_{1/n}^{\delta} \frac{dt}{t} \right] \\
&= o(1) + o \left[\frac{\log \log 1/t}{\log n} \right]_{1/n}^{\delta} + o \left[\frac{\log t}{\log n} \right]_{1/n}^{\delta} \\
&= o(1) + o \left[\frac{\log \log \delta^{-1}}{\log n} \right] + o \left[\frac{\log \log n}{\log n} \right] + o \left[\frac{\log \delta}{\log n} \right] + o \left[\frac{\log n^{-1}}{\log n} \right]
\end{aligned}$$

$$(3.1.5) = o(1).$$

Combining (3.1.3), (3.1.4) and (3.1.5), we see that (3.1.2) holds.

This completes the proof of theorem 1.

3.2. Proof of theorem 2.

Denoting by $\vec{\sigma}_n(x)$ the sum of the first n terms of the series (1.2.3) at a point $t=x$, we have

$$\begin{aligned}
\vec{\sigma}_n(x) &= \frac{-1}{\pi} \int_{-\pi}^{\pi} f(u) \frac{\partial}{\partial x} \left(\sum_{l=1}^n \sin v(u-x) \right) du \\
&= \frac{-1}{\pi} \int_0^{\pi} \frac{d}{dt} \left[\frac{\cos \frac{1}{2}t - \cos (n + \frac{1}{2})t}{2 \sin \frac{1}{2}t} \right] \{ f(x+t) + f(x-t) \} dt
\end{aligned}$$

$$= -\frac{1}{\pi} \int_0^{\pi} \frac{\cos \frac{1}{2} t - \cos (n + \frac{1}{2}) t}{2 \sin \frac{1}{2} t} dh(t).$$

Therefore

$$\begin{aligned} \bar{\sigma}_v(x) &= -\frac{1}{2\pi} \int_0^{\pi} [\cot \frac{1}{2} t (1 - \cos vt) + \sin vt] dh(t), \quad v < n, \\ &= -\frac{1}{2\pi} \left\{ \int_0^{1/n} + \int_{1/n}^{\pi} \right\} \cot \frac{1}{2} t (1 - \cos vt) dh(t) \\ &\quad - \frac{1}{2\pi} \int_0^{\pi} \sin vt dh(t) \\ &= -\frac{1}{2\pi} (P + Q + R), \end{aligned}$$

say.

But

$$\begin{aligned} |P| &\leq \int_0^{1/n} \left| \frac{\cos \frac{1}{2} t}{\sin \frac{1}{2} t} \cdot 2 \sin^2 \frac{vt}{2} \right| \cdot |dh(t)| \\ &\leq 2v \int_0^{1/n} |dh(t)| \\ &= 2v \chi(1/n) \\ &= o(1). \end{aligned}$$

and

$$\begin{aligned} -\frac{1}{2\pi} Q &= -\frac{1}{2\pi} \int_{1/n}^{\pi} \cot \frac{1}{2} t (1 - \cos vt) dh(t) \\ &= -\frac{1}{2\pi} \int_{1/n}^{\pi} \cot \frac{1}{2} t dh(t) + \frac{1}{2\pi} \int_{1/n}^{\pi} \cot \frac{1}{2} t \cos vt dh(t) \\ &= -\frac{1}{2\pi} \left[\cot \frac{1}{2} t h(t) \right]_{1/n}^{\pi} - \frac{1}{2\pi} \int_{1/n}^{\pi} \frac{1}{2} \operatorname{cosec}^2 \frac{1}{2} t h(t) dt \\ &\quad + \frac{1}{2\pi} \int_{1/n}^{\pi} \cot \frac{1}{2} t \cos vt dh(t) \\ &= \frac{1}{2\pi} \frac{\cos 1/2n \cdot 1/2n \cdot h(1/n)}{\sin 1/2n \cdot 1/2n} + H_n(x) + \frac{1}{2\pi} \int_{1/n}^{\pi} \cot \frac{1}{2} t \cos vt dh(t) \end{aligned}$$

$$= o(1) + H_n(x) + \frac{1}{2\pi} \int_{1/n}^{\pi} \cot \frac{1}{2}t \cos vt \, dh(t),$$

for $h(t)/t$ tends to zero with t , as $f'(x)$ exists.

Hence

$$\begin{aligned} \bar{\sigma}_v(x) - H_n(x) &= \frac{1}{2\pi} \int_{1/n}^{\pi} \cot \frac{1}{2}t \cos vt \, dh(t) \\ &\quad - \frac{1}{2\pi} \int_0^{\pi} \sin vt \, dh(t) + o(1) \\ &= \frac{1}{2\pi} \int_{1/n}^{\pi} \cot \frac{1}{2}t \cos vt \, dh(t) - \frac{1}{2\pi} \int_{1/n}^{\pi} \sin vt \, dh(t) \\ &\quad - \frac{1}{2\pi} \int_0^{1/n} \sin vt \, dh(t) + o(1) \\ &= \frac{1}{2\pi} \int_{1/n}^{\pi} \frac{\cos(v + \frac{1}{2})t}{\sin \frac{1}{2}t} \, dh(t) + o(1). \end{aligned}$$

Therefore

$$\bar{\sigma}_{n-k}(x) - H_n(x) = \frac{1}{2\pi} \int_{1/n}^{\pi} \frac{\cos(n - k + \frac{1}{2})t}{\sin \frac{1}{2}t} \, dh(t) + o(1).$$

Making use of the formula (1.1.4), we get

$$\begin{aligned} &\frac{1}{\log n} \sum_{k=0}^n \frac{1}{k+1} \left\{ \bar{\sigma}_{n-k}(x) - H_n(x) \right\} \\ &= \frac{1}{\log n} \sum_{k=0}^n \frac{1}{k+1} \left\{ \frac{1}{2\pi} \int_{1/n}^{\pi} \frac{\cos(n - k + \frac{1}{2})t}{\sin \frac{1}{2}t} \, dh(t) + o(1) \right\} \\ &= \int_{1/n}^{\pi} dh(t) \cdot \frac{1}{2\pi \log n} \sum_{k=0}^n \frac{1}{k+1} \frac{\cos(n - k + \frac{1}{2})t}{\sin \frac{1}{2}t} + o(1) \\ &= \int_{1/n}^{\pi} dh(t) \cdot \bar{N}_n(t) + o(1). \end{aligned}$$

In order to prove the theorem, we have to show that

$$(3.2.1) \quad \int_{1/n}^{\pi} dh(t) \cdot \bar{N}_n(t) = o(1),$$

as $n \rightarrow \infty$, when (2.2.2) exists.

We set

$$\begin{aligned} \int_{1/n}^{\pi} dh(t) \cdot \bar{N}_n(t) &= \left(\int_{1/n}^{\delta} + \int_{\delta}^{\pi} \right) dh(t) \cdot \bar{N}_n(t) \\ &= I_1 + I_2, \end{aligned}$$

say, where δ is small but fixed.

In order to prove $I_1 = o(1)$, we require a suitable estimate for the kernel $\bar{N}_n(t)$ in the interval (n^{-1}, δ) .

Since

$$\begin{aligned} \bar{N}_n(t) &= \frac{1}{2\pi \log n \sin \frac{1}{2}t} \left\{ \cos\left(n + \frac{3}{2}\right)t \sum_{k=0}^n \frac{\cos(k+1)t}{k+1} \right. \\ &\quad \left. + \sin\left(n + \frac{3}{2}\right)t \sum_{k=0}^n \frac{\sin(k+1)t}{k+1} \right\} \\ &= O \left[\frac{1}{t \log n} \left\{ 1 + \log 1/t \right\} \right], \end{aligned}$$

by (2.3.1) and (2.3.2), we have

$$\begin{aligned} I_1 &= \int_{1/n}^{\delta} dh(t) \cdot \bar{N}_n(t) \\ &= O \left(\int_{1/n}^{\delta} |dh(t)| \cdot |\bar{N}_n(t)| \right) \\ &= O \left(\frac{1}{\log n} \int_{1/n}^{\delta} |dh(t)| \cdot 1/t \right) + O \left(\frac{1}{\log n} \int_{1/n}^{\delta} |dh(t)| \cdot 1/t \cdot \log 1/t \right) \\ &= O \left[\frac{1}{\log n} \chi(t) \cdot 1/t \right]_{1/n}^{\delta} + O \left[\frac{1}{\log n} \int_{1/n}^{\delta} \chi(t) 1/t^2 dt \right] \end{aligned}$$

$$\begin{aligned}
& + O \left[\frac{1}{\log n} \chi(t) \cdot \frac{1}{t} \log \frac{1}{t} \right]_{1/n}^{\delta} + O \left[\frac{1}{\log n} \int_{1/n}^{\delta} \chi(t) \{ 1 + \log \frac{1}{t} \} \frac{dt}{t^2} \right] \\
& = O \left(\frac{1}{\log n} \right) + o \left(\frac{1}{(\log n)^2} \right) + o \left(\frac{1}{\log n} \right) + o \left[\frac{1}{\log n} \int_{1/n}^{\delta} \frac{dt}{t \log \frac{1}{t}} \right] \\
& \quad + o \left[\frac{1}{\log n} \int_{1/n}^{\delta} \frac{dt}{t} \right] \\
& = o(1) + o \left(\frac{\log \log \frac{1}{t}}{\log n} \right)_{1/n} + o \left(\frac{\log \frac{1}{t}}{\log n} \right)_{1/n}^{\delta}
\end{aligned}$$

$$(3.2.2) = o(1).$$

Finally, by virtue of Riemann-Lebesgue theorem and regularity of the method of summation, we have

$$I_2 = \int_{\delta}^{\pi} dh(t) \cdot \bar{N}_n(t)$$

$$(3.2.3) = o(1).$$

Combining (3.2.2) and (3.2.3), we see that (3.2.1) holds. This completes the proof of theorem 2.

4. It may be observed that there is a possibility of obtaining these theorems in an indirect manner by appealing to certain observations of Hille and Tamarkin [2]. For the harmonic summability of Fourier series, they had established a necessary and sufficient condition, *viz.*

A necessary and sufficient condition that the n -th harmonic mean of a Fourier series shall converge to $f(x)$ at a point x at which

$$\int_0^t |h(u)| du = o(t),$$

is that

$$(4.1) \quad \lim_{n \rightarrow \infty} \frac{1}{\log n} \int_{1/n}^{\delta} \sin nt \log \frac{1}{t} h(t) \frac{dt}{t} = 0.$$

They had suggested the possibility of deriving similar results for the summability of the derived and derived conjugate series. In view of the great complication involved in establishing their theorem (4.1), Iyengar [3] and Siddiqi [7] had obtained in a direct manner results pertaining to the harmonic summability of

Fourier series without making use of the theorem (4.1) of Hille and Tamarkin. There being no explicitly given, for derived and conjugate derived series, analogous theorems to (4.1), we have obtained results for these series in a direct manner.

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FINITE BENDING OF PLATES—III

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ABSTRACT

In parts I and II [4,5] the problems of bending of circular plates into spherical shells have been solved. In this paper the method has been extended to the problem of a rectangular plate bent into an elliptic shell. The case of a rectangular plate bent into a circular shell [1] has been obtained as a particular case.

1. INTRODUCTION

The problems of bending of plates into shells have been considered by B. R. Seth [1, 2, 3] on the basis of linear stress-strain relations,

$$\sigma_{ij} = \lambda \Delta + 2\mu \epsilon_{ij} \quad (1.1)$$

where ϵ_{ij} refer to the finite components of strain in the strained state of the body. In parts I and II [4, 5], we obtained solutions for the problems of bending of circular plates into spherical shells. In such problems apart from assuming tentative displacements, the calculation of expressions for ϵ_{ij} in terms of displacements, and their gradients in various coordinates, becomes very lengthy and complex [2, 4]. Obviously, the complexities will be more if the plates are to be bent into various other forms when the coordinate systems to be used are different curvilinear systems. In this paper it is shown that such problems can be attempted directly by using the expressions in terms of the metric tensors for strained and unstrained states of the body by introducing a curvilinear system of coordinates which move with body [6] and the problem of a rectangular block bent into an elliptic shell has been solved.

These results can be extended to the cases of plates bent into various other forms.

2. NOTATION AND FORMULAE

Following the notation of Green and Zerna [6], let x_i be the initial, y_i the final and θ_i the curvilinear, Coordinates of the strained body, so that

$$x_i = x_i(\theta_1, \theta_2, \theta_3) \text{ and } y_i = y_i(\theta_1, \theta_2, \theta_3) \quad (2.1)$$

The metric tensors g_{ij} and G_{ij} for the coordinate system θ_i in the unstrained and strained states respectively are given by

$$g_{ij} = \frac{\partial x^m}{\partial \theta^i} \frac{\partial x^m}{\partial \theta^j} \text{ and } G_{ij} = \frac{\partial y^m}{\partial \theta^i} \frac{\partial y^m}{\partial \theta^j} \quad (2.2)$$

Then the state of strain is measured by the symmetric covariant strain tensor

$$2 \gamma_{ij} = G_{ij} - g_{ij} \quad (2.3)$$

If T^{ij} is the contravariant stress tensor referred to θ_i coordinates in the strained body, the stress-strain relations (1.1) can be written as

$$\begin{aligned} T^{ij} &= \mu (G^{ir} G^{js} + G^{is} G^{jr} + \frac{2\eta}{1-2\eta} G^{ij} G^{rs}) \gamma_{rs} \\ &= 2\mu (\gamma^{ij} + \frac{\eta}{1-2\eta} G^{ij} \gamma^r_r) \end{aligned} \quad (2.4)$$

where η is the Poisson's ratio, so that $2\mu\eta / (1-2\eta) = \lambda$. The physical components of stress σ_{ij} and strain ϵ_{ij} are given by

$$\sigma_{ij} = (G_{ij} | G^{ii})^{\frac{1}{2}} T^{ij} \quad (2.5)$$

$$\epsilon_{ij} = (G^{ii} G^{jj})^{\frac{1}{2}} \gamma_{ij} \quad (2.6)$$

The equations of equilibrium in the absence of the body forces are

$$T^{ij} |_{,i} = 0 \text{ or } T^{ij}_{,i} + \Gamma^j_{ir} T^{ir} + \Gamma^i_{ir} T^{rj} = 0 \quad (2.7)$$

where vertical line denotes covariant differentiation and comma denotes differentiation with respect to θ_i in the strained body. The Christoffel symbols Γ^r_{ij} are given by

$$T^r_{ij} = \frac{G^{rs}}{2} (G^i_s, j + G^j_s, i - G^i_j, s) \quad (2.8)$$

3. BENDING OF A RECTANGULAR PLATE INTO AN ELLIPTIC CYLINDER

Let a rectangular plate in the undeformed state be bounded by the planes

$$x_1 = a_1, x_1 = a_2 \ (a_2 > a_1), x_2 = \pm b \text{ and } x_3 = \pm d.$$

Then it is bent into a part of elliptic cylinder whose internal and external boundaries are the confocal ellipses

$$x_1 = c \cosh \xi \cos \eta, x_2 = c \sinh \xi \sin \eta, \xi = \xi_i, i=1, 2 \quad (3.1)$$

and the edge $\eta = \pm \alpha$. Let y_i axes coincide with the x_i axes and the curvilinear coordinates θ_i in the deformed state be a system of orthogonal curvilinear coordinates (ξ, η, z) so that

$$y_1 = c \cosh \xi \cos \eta, y_2 = c \sinh \xi \sin \eta, y_3 = z \quad (3.2)$$

Since the deformation is symmetric about the x_1 axis, we see that

- (i) the planes $x_1 = \text{constant}$ in the undeformed state become the surfaces $\xi = \text{constant}$ in the deformed state.

(ii) the planes $x_2 = \text{constant}$ in the undeformed state become the surfaces $\eta = \text{constant}$ in the deformed state.

Then supposing no extension in the x_3 direction, we get

$$x_1 = f(\xi), x_2 = \varphi(\eta), x_3 = z \quad (3.3)$$

From (3.2) (3.3), and (2.2), the metric tensors for the strained and unstrained states are given by

$$G_{ij} = \begin{Bmatrix} c^2 (\cosh^2 \xi - \cos^2 \eta) & 0 & 0 \\ 0 & c^2 (\cosh^2 \xi - \cos^2 \eta) & 0 \\ 0 & 0 & 1 \end{Bmatrix} \quad (3.4)$$

$$g_{ij} = \begin{Bmatrix} f^{12} & 0 & 0 \\ 0 & \varphi^{12} & 0 \\ 0 & 0 & 1 \end{Bmatrix} \quad (3.5)$$

where $f^1 = \frac{df}{d\xi}$ and $\varphi^1 = \frac{d\varphi}{d\eta}$. Then

$$2\gamma_{ij} = \begin{Bmatrix} c^2 (\cosh^2 \xi - \cos^2 \eta) - f^{12} & 0 & 0 \\ 0 & c^2 (\cosh^2 \xi - \cos^2 \eta) - \varphi^{12} & 0 \\ 0 & 0 & 0 \end{Bmatrix} \quad (3.6)$$

Substituting (3.4) and (3.6) in (2.4) and (2.5), we get

$$\begin{aligned} 2\sigma_{11} &= 2(\lambda + \mu) - \frac{\lambda \varphi^{12}}{c^2 (\cosh^2 \xi - \cos^2 \eta)} - \frac{(\lambda + 2\mu) f^{12}}{c^2 (\cosh^2 \xi - \cos^2 \eta)} \\ 2\sigma_{22} &= 2(\lambda + \mu) - \frac{(\lambda + 2\mu) \varphi^{12}}{c^2 (\cosh^2 \xi - \cos^2 \eta)} - \frac{\lambda f^{12}}{c^2 (\cosh^2 \xi - \cos^2 \eta)} \\ 2\sigma_{33} &= \lambda \left[2 - \frac{f^{12} + \varphi^{12}}{c^2 (\cosh^2 \xi - \cos^2 \eta)} \right] \end{aligned} \quad (3.7)$$

From (2.7), (2.8) and (3.4) the equations of equilibrium to be satisfied are

$$T_{,1}^{11} + \frac{\sinh 2\xi}{2 (\cosh^2 \xi - \cos^2 \eta)} (3T^{11} - T^{22}) = 0 \quad (3.8)$$

$$T_{,2}^{22} + \frac{\sin 2\eta}{2 (\cosh^2 \xi - \cos^2 \eta)} (3T^{22} - T^{11}) = 0 \quad (3.9)$$

the third being identically satisfied.

On substituting (3.7) in (3.8) and (3.9) and solving, both these equations give the same solution

$$f^{1^2} + \varphi^{1^2} = A (\cosh^2 \xi - \cos^2 \eta)^{(2-2d)/2} \quad (3.10)$$

where $d = 2\mu / (\lambda + 2\mu)$ and A is an arbitrary constant.

Since f and φ are functions of ξ and η respectively, the equation (3.10) will be consistent only if η is small such that $\cos \eta$ can be replaced by unity and

$$\varphi = B \eta. \quad (3.11)$$

Then, we have

$$f^{1^2} = -B^2 + D (c \sinh \xi)^d \quad (3.12)$$

Substituting (3.11) and (3.12) in (3.7), we obtain

$$\begin{aligned} \sigma_{11} &= \frac{\mu}{d} \left[2-d + \frac{dB^2}{(c \sinh \xi)^2} - \frac{D}{(c \sinh \xi)^d} \right] \\ \sigma_{22} &= \frac{\mu}{d} \left[2-d - \frac{dB^2}{(c \sinh \xi)^2} - \frac{(1-d)D}{(c \sinh \xi)^d} \right] \\ \sigma_{33} &= \frac{\mu(1-d)}{d} \left[2 - \frac{D}{(c \sinh \xi)^d} \right] \end{aligned} \quad (3.13)$$

4. BOUNDARY CONDITIONS

If the plate is bent by applying force to the edges only, we must have

$$\sigma_{11} = 0 \text{ when } \xi = \xi_1, \quad i = 1, 2 \quad (4.1)$$

which gives

$$2-d + \frac{dB^2}{(c \sinh \xi_i)^2} - \frac{D}{(c \sinh \xi_i)^d} = 0 \quad i = 1, 2 \quad (4.2)$$

Solving these equations, we obtain

$$B^2 = \frac{2-d}{d} \frac{(c \sinh \xi_2)^d - (c \sinh \xi_1)^d}{(c \sinh \xi_2)^{2-d} - (c \sinh \xi_1)^{2-d}} (c \sinh \xi_1 c \sinh \xi_2)^{2-d} \quad (4.3)$$

$$D = (2-d) \frac{(c \sinh \xi_2)^2 - (c \sinh \xi_1)^2}{(c \sinh \xi_2)^{2-d} - (c \sinh \xi_1)^{2-d}} \quad (4.4)$$

On the straight edges $\eta = \pm \alpha$, the distribution of tractions gives rise to a force F_1 and a couple M_1 given by

$$F_1 = \int_{\xi_1}^{\xi_2} \sigma_{22} c \sinh \xi d\xi \quad (4.5)$$

$$M_1 = \int_{\xi_1}^{\xi_2} \sigma_{22} c \sinh \xi c \cosh \xi d\xi. \quad (4.6)$$

The force F_2 required to keep the length of the plate constant in the direction of the axis of the cylinder and the couple M_2 in the axial plane applied to the plane ends of the cylinder per unit are between η and $\eta + \delta\eta$ are given by

$$F_2 = \int_{\xi_1}^{\xi_2} \sigma_{33} c^2 \sinh^2 \xi d\xi \quad (4.7)$$

$$- M_2 = \int_{\xi_1}^{\xi_2} \sigma_{33} c^2 \sinh^2 \xi c \cosh \xi d\xi \quad (4.8)$$

Substituting (3.13) in (4.5) to (4.8), we get

$$F_1 = \frac{\mu}{d} \left[(2-d) (c \cosh \xi_2 - c \cosh \xi_1) - \frac{dB^2}{c} \log \frac{\tanh \xi_2/2}{\tanh \xi_1/2} - (1-d)D \int_{\xi_1}^{\xi_2} (c \sinh \xi)^{1-d} d\xi \right] \quad (4.9)$$

$$M_1 = \mu \left[\frac{c \sinh^2 \xi_2 - c \sinh^2 \xi_1}{2} - \frac{(2-d) \log \frac{c \sinh \xi_2}{c \sinh \xi_1} (c \sinh \xi_1 c \sinh \xi_2)^{2-d} \{ (c \sinh \xi_2)^d - (c \sinh \xi_1)^d \}}{d \{ (c \sinh \xi_2)^{2-d} - (c \sinh \xi_1)^{2-d} \}} \right] \quad (4.10)$$

$$F_2 = \frac{\mu (1-d)}{d} \left[\frac{c^2 (\sinh 2\xi_2 - \sinh 2\xi_1)}{2} - c^2 (\xi_2 - \xi_1) - D \int_{\xi_1}^{\xi_2} (c \sinh \xi)^{2-d} d\xi \right] \quad (4.11)$$

$$-M_2 = \lambda \left[\frac{(c \sinh \xi_2)^3 - (c \sinh \xi_1)^3}{3} + \frac{2-d}{3-d} \{ (c \sinh \xi_2)^2 - (c \sinh \xi_1)^2 \} \frac{(c \sinh \xi_2)^{3-d} - (c \sinh \xi_1)^{3-d}}{(c \sinh \xi_2)^{2-d} - (c \sinh \xi_1)^{2-d}} \right] \quad (4.12)$$

6. PARTICULAR CASE

When $c \cosh \xi = c \sinh \xi$ in (3.1), we get the case of a rectangular plate bent into a part of right circular cylinder of inner and outer radii γ_i , $i=1, 2$ respectively, so that [7] $c \rightarrow 0$, $\xi \rightarrow \infty$ and consequently

$$c \cosh \xi \text{ and } c \sinh \xi \rightarrow \gamma \quad (5.1)$$

Then the curvilinear system (ξ, η, z) is replaced by the cylindrical coordinate (γ, η, z) . Then the equations (4.10) and (4.12) reduce to

$$M_1 = \mu \left[\frac{\gamma_2^3 - \gamma_1^3}{2} - \frac{(2-d)(\gamma_1 \gamma_2)^{2-d} (\gamma_2^d - \gamma_1^d) \log \gamma_2 / \gamma_1}{d(\gamma_2^{2-d} - \gamma_1^{2-d})} \right] \quad (5.2)$$

$$-M_2 = \lambda \left[\frac{\gamma_2^3 - \gamma_1^3}{3} + \frac{2-d}{3-d} \frac{(\gamma_2^2 - \gamma_1^2)(\gamma_2^{3-d} - \gamma_1^{3-d})}{\gamma_2^{2-d} - \gamma_1^{2-d}} \right] \quad (5.3)$$

which tally with the previous result [1].

In this case, the equation of equilibrium (3.8) is replaced by

$$\frac{\partial \sigma_{11}}{\partial \gamma} + \frac{\sigma_{11} - \sigma_{22}}{\gamma} = 0 \quad (5.4)$$

Therefore

$$F_1 = \int_{\gamma_1}^{\gamma_2} \sigma_{22} d\gamma = \int_{\gamma_1}^{\gamma_2} \frac{d}{d\gamma} (\gamma \sigma_{11}) d\gamma = (\gamma \sigma_{11})_{\gamma_1}^{\gamma_2} = 0 \quad (5.5)$$

and from (5.1), and (4.7), it can be readily seen that F_2 also becomes zero.

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A STUDY ON THE FORMATION OF COMPLEX COMPOUNDS BETWEEN BIVALENT AND UNIVALENT SALTS

PART VI. SYSTEMS : $\text{HgI}_2\text{—NaCN—H}_2\text{O}$ and $\text{Hg(CN)}_2\text{—MI—H}_2\text{O}$ (M = Na or Li)

By

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ABSTRACT

A preparative study of the systems: $\text{HgI}_2\text{—NaCN—H}_2\text{O}$ and $\text{Hg(CN)}_2\text{—MI—H}_2\text{O}$ (M = Na or Li) has been carried out and complexes of the type $\text{MHg(CN)}_2\text{I}$ have been isolated and shown to remain unchanged on recrystallisation.

Extensive work has been done on the mercury halide and pseudohalide complexes of the type $(\text{HgX}_3)^-$ and $(\text{HgX}_4)^{2-}$ (X = Cl, Br, I, CN or CNS) but the mixed pseudohalides and halide-pseudohalide complexes of mercury although studied earlier¹⁻⁴, have received considerable attention only for the last few years⁵⁻⁶. The authors⁷⁻¹⁰ have recently carried out a preparative study of the systems: $\text{HgX}_2\text{—MCN—H}_2\text{O}$ (M = K or Na and X = Cl, Br or CNS) and $\text{Hg(CN)}_2\text{—MX—H}_2\text{O}$ (M = K, Na, NH_4 or Li and X = Cl, Br or CNS) and have isolated three different types of complexes MHgX_2CN , $\text{MHg(CN)}_2\text{X}$ and $\text{M}_2\text{Hg(CN)}_4$ from the potassium systems whereas from all other systems the only compound which could be isolated corresponded to the formula $\text{MHg(CN)}_2\text{X}$. In this investigation, the above study has been extended to the systems: $\text{HgI}_2\text{—NaCN—H}_2\text{O}$ and $\text{Hg(CN)}_2\text{—MI—H}_2\text{O}$ (M = Na or Li) and complexes of the type $\text{MHg(CN)}_2\text{I}$ have been isolated. The stable nature of these iodide complexes conforms to the general order of stability already pointed out in Part III⁹ of the series.

EXPERIMENTAL

Materials. B. D. H. and E. Merck chemicals (Laboratory Reagent) were used for the preparative work.

Analytical Methods: Mercury was estimated as sulfide without adding the acid as the addition of acid precipitated mercuric iodide. Iodide was estimated by holding cyanide in solution with formaldehyde and precipitating with silver nitrate. The nitric acid was added after adding silver nitrate as the addition of acid precipitates mercuric iodide. Cyanide was estimated utilising the observation of Aggarwal and Mehrotra⁵ that in mixtures of mercuric iodide and cyanide with potassium cyanide and iodide respectively, cyanide ions corresponding to the formation of Hg(CN)_2 remained unionised and the remaining cyanide ions along with any iodide ions got quantitatively precipitated as insoluble silver salt by holding cyanide in solution with formaldehyde and a second estimation was made without adding formaldehyde which gave iodide and any ionisable cyanide. From the two results, the amount of total cyanide was computed.

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PREPARATIVE EXPERIMENTS AND RESULTS

Products obtained from reaction mixtures having different molecular ratio of the reactants, namely HgI_2 and NaCN or $\text{Hg}(\text{CN})_2$ and NaI or LiI in concentrated aqueous solutions which were filtered to remove any turbidity if obtained, have been analysed. In all cases, the compound isolated corresponds to the general formula $\text{MHg}(\text{CN})_2\text{I}$. The compounds crystallise in the form of white needles which turn yellow on standing, are soluble in water and deposit red mercuric iodide on acidification. In the table below are summarised the analyses of various products obtained from different molar ratio of the reactants and thoroughly dried at 110°C .

Molar ratio of the reaction mixture	Analysis of the product			Yield
	Hg	CN	I	
1. $0.8\text{HgI}_2 \cdot \text{NaCN}$	49.9	13.0	31.3	55.0%
2. $\text{HgI}_2 \cdot 2\text{NaCN}$	49.6	12.8	29.0	34.0%
3. $\text{HgI}_2 \cdot 4\text{NaCN}$	49.7	12.9	29.0	20.0%
4. $\text{Hg}(\text{CN})_2 \cdot \text{NaI}$	48.5	12.6	28.5	18.5%
5. $\text{Hg}(\text{CN})_2 \cdot 2\text{NaI}$	49.3	12.8	29.2	38.0%
6. $\text{Hg}(\text{CN})_2 \cdot 4\text{NaI}$	49.3	12.8	29.4	35.0%
$\text{NaHg}(\text{CN})_2\text{I}$ requires :	49.7	12.9	31.5	
7. $\text{Hg}(\text{CN})_2 \cdot \text{LiI}$	50.9	13.3	31.3	12.0%
8. $\text{Hg}(\text{CN})_2 \cdot 2\text{LiI}$	50.4	13.2	30.5	10.5%
9. $\text{Hg}(\text{CN})_2 \cdot 4\text{LiI}$	51.2	13.3	31.2	68.0%
$\text{LiHg}(\text{CN})_2\text{I}$ requires :	51.8	13.5	32.9	

The above compounds did not change in composition on recrystallisation showing that they are quite stable.

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*More HgI_2 could not be dissolved.

EFFECT OF SOIL MOISTURE CONDITIONS AND GIBBERELLIN APPLICATION ON RICE PRODUCTION - II

By

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ABSTRACT

Different concentrations of gibberellin were sprayed on moisture depleted as well as on normally irrigated plants and their effect on the yield character was noted. The main findings and conclusions derived are as under :—

- (1) In a late maturing variety, depletion of soil moisture after grain formation stage reduces the yield of both grain and straw considerably.
- (2) 50 ppm. has been found to be the best concentration of gibberellin for application on normally irrigated as well as on moisture depleted plants.
- (3) The hormone should only be applied if there are adequate supply of available soil moisture and nutrients for the plant.

INTRODUCTION

Rice is a semi-aquatic plant and requires an abundant supply of water for its growth. Though India possesses the largest irrigation system in the world still bulk of the crop is mostly dependent on the vagaries of the monsoon. Therefore, an efficient use of water for maximum production has got special importance, but unfortunately in our country very little information is available about it. Experiments conducted in Madras and Andhra Pradesh (as cited by Ghose *et al*, 1960) have shown that by intermittent drying and irrigating the field every third day, a crop yield of the same order may be obtained as by maintaining continuously a three inch depth of water in the field. The authors (1962) found that progressively depleted soil moisture conditions just after the grain formation stage result in higher production of both grain and straw.

It is well known that in the eastern part of U. P. bulk of precipitation occurs in between middle of June to the end of September, and therefore, one method of avoiding drought injury had been to grow such early maturing varieties of paddy which complete their life cycles before the withdrawal of the monsoon. But, as the population of our country is increasing we require higher yield per unit area and hence cannot depend too long on low yielding short duration varieties.

Scientists all over the world, apart from introducing new promising strains of crops, better manurial and cultural practices and other improvements, are also trying certain growth promoting substances, popularly known as plant hormones to achieve quick results. Amongst the various growth promoting substances, gibberellin is of recent origin. Stowe and Yamaki (1957) have dealt with the history and physiological action of gibberellic acid on various plants. Brian and Grove (1957) have discussed the general effect of gibberellic acid on different types of plant. Some

of the important findings and conclusions largely derived from their work are as below :

- (1) It promotes growth in a number of species, e. g., when wheat plant was treated, the mean height of the flowering tillers was increased considerably and the ears emerged from the sheathing leaves a few days before than those of untreated plants. However, there was a reduction in the number of flowering tillers and the ear per tiller.
- (2) It induces flowering in biennial plants without need for a cold treatment.
- (3) It breaks dormancy.

A number of workers such as Nieman and Bernstein (1959), Krishnamurthi *et al* (1959), Singh and Randhawa (1959), Nagrutskaa (1961) and Avdonin and Yu (1961) have obtained higher yield of fruits in a number of plants.

To the best of knowledge of the authors, no work has been reported so far on the relationship of soil moisture (during the later stages of plant growth and gibberellin application. This paper deals with the investigations carried on the rice crop over this important aspect.

EXPERIMENTAL METHOD

This experiment was laid out in medium sized cement pots which were filled with 20 Kg. of well prepared surface soil of a field where no manurial or any other type of experiments were conducted for the last 5 years. A mid-duration variety of paddy (T_{21}) was selected and the seedlings were raised under the natural conditions of the farm and were allowed to grow for 6 weeks. Soil of each pot was puddled in the last week of July 1961 after adding equal and measured amount of water and three seedlings of uniform growth and appearance were transplanted in each of them. Plants were provided with liberal but measured amount of water and were timely intercultured and dusted with gammexane. They were allowed to grow upto 90 days (grain formation stage) and then they were brought inside a glass house so that rain water may not interfere during the period of experimentation.

Pots were now divided in two sets, one receiving liberal but measured amount of water till the end of the life-cycle and denoted as I_n , and in case of the other, fresh supply of irrigation water was stopped after keeping 2" of standing water at initial stage and was denoted by I_1 . Following concentrations of gibberellin were sprayed on the plants of each set at an interval of 15 days, starting just after the transplanting and continued till the harvest time :

- (i) G_0 - Control, i.e., no spraying of the hormone.
- (ii) G_1 - 25 p. p. m.
- (iii) G_2 - 50 p. p. m. and
- (iv) G_3 - 100 p. p. m.

Pots under the following 8 treatment combinations were arranged in a factorial block design with three replications :

- | | | | |
|-----------------|-----------------|-----------------|-----------------|
| (1) $I_n G_0$, | (2) $I_n G_1$, | (3) $I_n G_2$, | (4) $I_n G_3$, |
| (5) $I_1 G_0$, | (6) $I_1 G_1$, | (7) $I_1 G_2$, | (8) $I_1 G_3$. |

Harvesting of the plants was done when their grains attained full maturity. The yield data was treated statistically by the method described by Goulden (1952) to find out the level of significance between the treatments.

EXPERIMENTAL FINDINGS

Yield of grain and straw per plant has been presented in the Table 1. Tables 2 and 3 are concerned with analysis of variance and interactions due to different treatments.

TABLE 1

Effect of Soil Moisture Conditions and Gibberellin Application on the Yield (gm. per plant) of grain and Straw.

Treatments	Replications			Mean
	I	II	III	
GRAIN				
(1) $I_n G_0$	14.8	17.6	18.0	16.8
(2) $I_n G_1$	15.4	20.1	19.7	18.4
(3) $I_n G_2$	14.5	19.1	18.9	17.5
(4) $I_n G_3$	12.4	17.2	16.6	15.4
(5) $I_1 G_0$	11.6	15.2	14.9	13.9
(6) $I_1 G_1$	9.5	14.1	13.6	12.4
(7) $I_1 G_2$	11.6	17.8	17.7	15.7
(8) $I_1 G_3$	13.0	19.0	19.0	17.0
STRAW				
(1) $I_n G_0$	61.0	52.9	59.8	57.9
(2) $I_n G_1$	65.0	58.6	64.8	62.8
(3) $I_n G_2$	68.0	58.0	66.0	64.0
(4) $I_n G_3$	64.2	55.6	62.9	60.9
(5) $I_1 G_0$	46.7	40.3	46.5	45.5
(6) $I_1 G_1$	63.7	55.0	62.8	60.5
(7) $I_1 G_2$	64.9	57.2	63.0	61.7
(8) $I_1 G_3$	52.6	45.1	51.3	49.8

TABLE 2
Analysis of Variance

Due to	D. F.	M. S. for grain	M. S. for straw
Replication	2	55.45	148.04
Irrigation (I)	1	31.06**	316.82**
Gibberellin (G)	3	5.27**	179.60**
I × G	3	11.75**	50.69**
Error	14	0.42	0.42
Total	23	103.95	695.57

** Significant at 1% level.

TABLE 3
Interaction Between Soil Moisture and Gibberellin

Irrigation	Gibberellin				Mean	C. D.
	G ₀	G ₁	G ₂	G ₃		
GRAIN						
I _n	16.8	18.4	17.5	15.4	17.0	I = 0.556
I ₁	13.9	12.4	15.7	17.0	14.8	G = 0.792
Mean	15.4	15.4	16.6	16.2		I × G = 1.134
STRAW						
I _n	57.9	62.8	64.0	60.9	61.4	I = 0.556
I ₁	44.5	60.5	61.7	49.8	54.1	G = 0.792
Mean	51.2	61.7	62.9	55.9		I × G = 1.134

A study of the Table 2 clearly shows that the yield of grain and straw is highly influenced by soil moisture, gibberellin and their interactions.

Data presented in Table 3 clearly show that depleted soil moisture affects the grain and straw yield very adversely. Taking the effect of gibberellin irrespective of soil moisture, it was found that higher concentrations generally favour higher production of both grain and straw and 50 p. p. m. application appeared to be the most effective. Grain yield also improved by 100 p. p. m. application, but this concentration was not found so useful for straw production. Although grain yield did not increase with 25 p. p. m. application, but straw yield improved so well that the performance was next only to 50 p. p. m. application.

Considering the combined effect of soil moisture and gibberellin, it was noted that normally irrigated plant produced the highest amount of grain when sprayed with 25 p. p. m solution. Further, in case of grain, although the performance due to 50 p. p. m was slightly less responsive than that due to 25 p. p. m. application, the order was reversed in the matter of straw production. As regards the grain production in case of moisture depleted plants, highly improved result was obtained where 100 p. p. m. followed by 50 p. p. m. concentrations were used. Straw yield on the other hand was highly increased where plants were treated with 50 or 25 p. p. m. concentrations.

DISCUSSION AND CONCLUSIONS

One of the striking features of the investigation is that depleted soil moisture reduces the yield of grain and straw very significantly. This is quite reasonable in view of the fact that the requirement of moisture is greater in a mid-duration variety than the early maturing one, because the former takes a longer period in completing its life-cycle than the latter. This increasingly low availability of moisture might have caused adverse effect on the root hairs and ultimately on the plant. Stocker (1930) has also stated that root hairs of growing plants on the steppes of Hungary died when the pull of the soil approached 30 atmospheres. Shull (1916) has stated that plant usually wilts when the rate of movement of water from particle to particle of soil and so to the root hairs falls below the necessary requirement to replace that has been lost from the aerial parts even under conditions of low water loss from the plant. This low status of moisture in the plant might have led to lesser enzymatic activities seated in the protoplasm, thereby not only causing reduction in growth but also in translocation of carbohydrate, protein, oil, fat etc., to the grains and ultimately lowering of the yield. This result is entirely contrary to our previous findings in which it was observed that progressively depleted soil moisture after grain formation stage and potassium application induce higher grain and straw yield. But that experiment was conducted on a short duration variety *viž.*, N₂₂ and hence among the various causes responsible for lowering the yield in the present investigation, variety also might be one of them.

The other important finding is that, in general, application of gibberellin on moisture depleted plants reduces the grain and straw yield to a great extent. A number of workers such as Chuvashina (1960), Evtushenko and Gonchar (1960), Kushizaki and Hoshi (1961), Evtushenko (1961), and Lyashchenko (1961) have shown that application of gibberellin causes more shoot growth than the root. During the course of the present investigation also it was observed that the fresh and dry weight of roots of the gibberellin treated plants was lower than the untreated ones. This inhibited root growth, owing to less coverage of the moisture absorbing area, probably failed to meet the increasing demand of moisture and other soluble nutrients for the growing shoot. It is also possible that as the application of gibberellin reduces the osmotic pressure in the plant (Yoda, 1961), it leads to lowering of moisture absorption rate still further and causes adverse effect on a number of processes going on in the tissues of the plant. On the other hand, normally irrigated and gibberellin treated plants, in spite of their inhibited root growth, did not face such adverse condition of low moisture supply, and kept up their normal growth and development and hence yielded better.

Hence it appears that better performance, particularly in depleted soil moisture condition, can be obtained if gibberellin is applied on an early maturing variety having extensive root system. It is also suggested that gibberellin should be applied only when adequate irrigation and nutrients for plant growth are available.

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SUMMARY

The chief objective of undertaking the present investigation was to find out how far soil moisture conditions and gibberellin application affect the yield of rice. The main findings and conclusions may be stated as below :

(1) Depleted soil moisture after grain formation stage is harmful to a longer duration variety.

(2) Application of gibberellin with 50 p. p. m. concentration has been found to be most useful both on the normally irrigated and moisture depleted plants, and

(3) Gibberellin should only be applied if there are adequate supply of available soil moisture and nutrients to the plant.

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SOME OSCILLATIONS OF AN INFINITE CYLINDER WITH A MAGNETIC FIELD

By

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ABSTRACT

In this paper the oscillations of an infinite gaseous cylinder with a mathematically tractable magnetic field are discussed using virial tensor equations. It gives nine modes which can be classified into three groups. First group consists of three neutral modes. The second group is of four non-radial modes. These are two different modes, each of multiplicity two. In the third group the two modes are coupled.

1. INTRODUCTION

Chandrasekhar (1960) has obtained the tensor form of virial theorem for small oscillations about equilibrium for a fluid configuration in which a magnetic field prevails. Using this theorem a general characteristic equation of ninth order for λ^2 (square of frequency of pulsations) has been obtained here for an axially symmetric configuration with a magnetic field. In this note we have studied these nine modes of oscillations for the case of an infinite cylinder. The pulsations for the case of no magnetic field have also been discussed.

2. The Virial Theorem for Small Oscillations.

Consider an inviscid and infinitely conducting fluid in which a magnetic field \vec{H} prevails. Then the virial theorem gives (Chandrasekhar, 1960)

$$\frac{d}{dt} \int_V \rho x_k u_i dv = 2T_{ki} - P_{ki} + \delta_{ki} \int_V p dv + \delta_{ki} M - 2M_{ki}, \quad (1)$$

where T_{ki} , P_{ki} and M_{ki} are the tensors defining kinetic energy, potential energy and magnetic energy respectively. These are given by

$$T_{ki} = \frac{1}{2} \int_V \rho u_k u_i dv, \quad (2)$$

$$P_{ki} = -\frac{1}{2} \int_V \rho V_{ki} dv, \quad (3)$$

$$M_{ki} = 1/8\pi \int_V H_k H_i dv \quad (4)$$

In equation (3), V_{ki} is Chandrasekhar's potential tensor and the other symbols

have their usual meanings.

If $\vec{\xi}(x, t)$ denotes the Lagrangian displacement of an element of mass from its position of equilibrium, then the perturbation form of the virial theorem can be written from (1) as (Chandrasekhar, 1960)

$$\begin{aligned} \frac{d^2}{dt^2} \int_V \rho x_k \xi_i dv = & - \int_V P_{ij} \frac{\partial V_{ki}}{\partial x_j} dv - (\gamma - 1) \delta_{ki} \int_V p \operatorname{div} \vec{\xi} dv \\ & - \frac{1}{4\pi} \int_V \left\{ H_j \left(H_i \frac{\partial \xi_k}{\partial x_j} + H_k \frac{\partial \xi_i}{\partial x_j} \right) - H_k H_i \frac{\partial \xi_j}{\partial x_j} \right\} dv \\ & + \delta_{ki} \left\{ \frac{1}{8\pi} \int_V \left(2 H_j H_m \frac{\partial \xi_m}{\partial x_j} - |H|^2 \frac{\partial \xi_j}{\partial x_j} \right) dv \right\}. \end{aligned} \quad (5)$$

This equation gives the tensor form of virial theorem for small perturbations about equilibrium state of the configuration in which the magnetic field \vec{H} prevails. We take the equilibrium configuration to be one having symmetry about the three coordinate planes. Chandrasekhar and Lebovitz (1962) have discussed the symmetry of the tensor P_{ki} . We study here the symmetry of the tensor M_{ki} .

Due to the symmetry mentioned above in ρ , P_{ki} is diagonal. Therefore, from the equation of equilibrium (putting $u_i = 0$ in equation (1))

$$P_{ki} - 2M_{ki} = - \delta_{ki} \left(\int_V p dv + M \right). \quad (6)$$

This gives

$$M_{ki} = 0 \quad (k \neq i) \quad (7)$$

Therefore we conclude that M_{ki} is also diagonal.

3. Characteristic Equation For λ^2 .

To discuss the oscillations of the above configuration we assume the following general form of "trial function" for $\vec{\xi}$

$$\xi_j = X_{jr} x_r e^{\lambda t}, \quad (8)$$

where X_{jr} are constants and λ is the frequency of pulsations. Substituting (8) in (5), we get

$$\begin{aligned} \lambda^2 X_{im} I_{mk} = & J X_{jj} \delta_{ki} - X_{jm} P_{mj; ki} \\ & - 2 (X_{kj} M_{ji} + M_{jk} X_{ij} - M_{ki} X_{jj}) \\ & - \delta_{ki} (M X_{jj} - 2 X_{pj} M_{jp}), \end{aligned} \quad (9)$$

where

$$P_{mj; ki} = \int_v \rho x_m \partial v_{ki} \partial x_j dv, \quad (10)$$

$$J = -(\gamma - 1) \int_v p dv, \quad (11)$$

and

$$I_{mk} = \int_v \rho x_m x_k dv. \quad (12)$$

We suppose that the configuration has axial symmetry about the x_3 -axis and also that x_1 - x_2 plane is a plane of symmetry. This includes the triplanar symmetry. Therefore the relations derived before still hold good.

This gives

$$P_{11} = P_{22}, \quad I_{11} = I_{22}. \quad (13)$$

Also, we have, from equation (6) by putting $i = k$,

$$(P_{11} + M) - 2M_{11} = (P_{22} + M) - 2M_{22} = (P_{33} + M) - 2M_{33} = - \int_v p dv \quad (14)$$

The first two of these relations give

$$M_{11} = M_{22} \quad (15)$$

From (14) and (15)

$$J = (\gamma - 1) (P_{33} + 2M_{11} - M_{33}). \quad (16)$$

The nine equations given by the tensor equation (9) are

$$\begin{aligned} \lambda^2 X_{11} I_{11} = & J X_{jj} - (X_{11} P_{11; 11} + X_{22} P_{22; 11} + X_{33} P_{33; 11}) \\ & - 2M_{11} (X_{11} - X_{22}) - M_{33} (X_{11} + X_{22} - X_{33}), \end{aligned} \quad (17)$$

$$\lambda^2 X_{22} I_{11} = J X_{jj} - (X_{11} P_{22;11} + X_{22} P_{11;11} + X_{33} P_{33;11}) - 2M_{11} (X_{22} - X_{11}) - M_{33} (X_{11} + X_{22} - X_{33}), \quad (18)$$

$$\lambda^2 X_{33} I_{33} = J X_{jj} - (X_{11} + X_{22}) P_{33;11} - X_{33} P_{33;33} - 2M_{11} X_{33} + M_{33} (X_{11} + X_{22} - X_{33}) \quad (19)$$

$$\lambda^2 X_{21} I_{11} = -(X_{21} + X_{12}) P_{12;12} - 2M_{11} (X_{12} + X_{21}), \quad (20)$$

$$\lambda^2 X_{12} I_{11} = -(X_{12} + X_{21}) P_{12;12} - 2M_{11} (X_{12} + X_{21}), \quad (21)$$

$$\lambda^2 X_{31} I_{11} = -X_{13} P_{31;13} - X_{31} P_{13;13} - 2(M_{11} X_{31} + M_{33} X_{13}), \quad (22)$$

$$\lambda^2 X_{13} I_{33} = -X_{13} P_{31;13} - X_{31} P_{13;13} - 2(M_{11} X_{31} + M_{33} X_{13}), \quad (23)$$

$$\lambda^2 X_{23} I_{33} = -X_{23} P_{31;13} - X_{32} P_{13;13} - 2(M_{11} X_{32} + M_{33} X_{23}), \quad (24)$$

$$\lambda^2 X_{32} I_{11} = -X_{23} P_{31;13} - X_{32} P_{13;13} - 2(M_{11} X_{32} + M_{33} X_{23}). \quad (25)$$

In the following discussion, we have taken (Chandrasekhar and Lebovitz, 1962).

$$\left. \begin{aligned} P_{11;11} &= P_{22;22} = A \\ P_{11;22} &= P_{22;11} = B \\ P_{11;33} &= P_{22;33} = P_{33;11} = P_{33;22} = C \\ P_{33;33} &= D \\ P_{12;12} &= P_{21;21} = \frac{1}{2} (A - B) \\ P_{13;13} &= P_{23;23} = C - P_{11} \\ P_{31;13} &= P_{32;23} = C - P_{33} \end{aligned} \right\} \quad (26)$$

where A, B, C and D are connected by the following three relations

$$A + B + C = P_{11}, \quad (27)$$

$$2C + D = P_{33}, \quad (28)$$

$$3A - B + 2C = 0. \quad (29)$$

4. THE MODES OF OSCILLATIONS OF CYLINDRICAL GASEOUS MASS

In this section the modes of oscillations of gaseous cylindrical mass in the presence of magnetic field satisfying the condition (16), are discussed.

For the case of cylinder, we have

$$I_{11} = I_{22} = I/4, \quad I_{33} = I/2. \quad (30)$$

In this case, equations (22) to (25) reduce to the following forms:

$$\lambda^2 X_{31} I/4 = -X_{13} P_{31;13} - X_{31} P_{13;13} - 2(M_{11}X_{31} + M_{33}X_{13}), \quad (31)$$

$$\lambda^2 X_{13} I/4 = -X_{13} P_{31;13} - X_{31} P_{13;13} - 2(M_{11}X_{31} + M_{33}X_{13}), \quad (32)$$

$$\lambda^2 X_{23} I/4 = -X_{23} P_{31;13} - X_{32} P_{13;13} - 2(M_{11}X_{32} + M_{33}X_{23}), \quad (33)$$

$$\lambda^2 X_{32} I/4 = -X_{23} P_{31;13} - X_{32} P_{13;13} - 2(M_{11}X_{32} + M_{33}X_{23}) \quad (34)$$

Subtracting (32) from (31), we get

$$\lambda^2 (X_{31} - 2X_{13}) I/4 = 0, \quad (35)$$

which gives

$$\lambda^2 = 0 \text{ (if } X_{31} \neq 2X_{13}) \quad (36)$$

Adding (31) and (32), we get

$$\lambda^2 (X_{31} + 2X_{13}) I/4 = -2X_{13} P_{31;13} - 2X_{31} P_{13;13} - 4(M_{11}X_{31} + M_{33}X_{13}) \quad (37)$$

This gives

$$\begin{aligned} \lambda^2 &= -2/I [3C - 2P_{11} - P_{33} + 4M_{11} + 2M_{33}], \\ &= 2/I [P - 2M - 3C] \text{ (if } X_{31} = 2X_{13}). \end{aligned} \quad (38)$$

In equation (38), we have used (26)

Similarly equations (33) and (34) give

$$\lambda^2 = 0 \text{ (if } X_{32} \neq 2X_{23}), \quad (39)$$

and

$$\begin{aligned} \lambda^2 &= -2/I [3C - 2P_{11} - P_{33} + 4M_{11} + 2M_{33}], \\ &= 2/I [P - 2M - 3C] \text{ (if } X_{32} = 2X_{23}) \end{aligned} \quad (40)$$

Also from equations (20) and (21), on subtracting and adding, we get

$$\lambda^2 (X_{21} - X_{12}) I/4 = 0, \quad (41)$$

which gives

$$\lambda^2 = 0 \text{ (if } X_{12} \neq X_{21}), \quad (42)$$

and

$$\lambda^2 (X_{21} + X_{12}) I/4 = -2 (X_{12} + X_{21}) P_{12;12} - 4M_{11} (X_{12} + X_{21}). \quad (43)$$

This gives

$$\lambda^2 = 4/I (B - A - 4M_{11}) \quad (\text{if } X_{12} \neq -X_{21}). \quad (44)$$

Next we consider equations (17) to (19).

Subtracting equation (18) from (17), we have

$$\lambda^2 (X_{11} - X_{22}) I/4 = (B - A - 4M_{11}) (X_{11} - X_{22}), \quad (45)$$

which gives

$$\lambda^2 = 4/I (B - A - 4M_{11}) \quad (\text{if } X_{11} \neq X_{22}). \quad (46)$$

Adding (17) and (18) and subtracting twice equation (19) from it, we get

$$\begin{aligned} & (\lambda^2 I/4 + A + B - 2C + 4M_{33}) (X_{11} + X_{22}) \\ & + X_{33} (-\lambda^2 I + 2C - 2D - 4M_{11} - 4M_{33}) = 0. \end{aligned} \quad (47)$$

Equations (47) and (19) can be written in the matrix notation as

$$\begin{vmatrix} \lambda^2 I/4 + \alpha_1 & -\lambda^2 I + \alpha_2 \\ -J + C - M_{33} & \lambda^2 I/2 + \alpha_3 \end{vmatrix} \begin{vmatrix} X_{11} + X_{22} \\ X_{33} \end{vmatrix} = 0, \quad (48)$$

where

$$\left. \begin{aligned} \alpha_1 &= A + B - 2C + 4M_{33} \\ \alpha_2 &= 2C - 2D - 4M_{11} - 4M_{33} \\ \alpha_3 &= -J + D + M_{33} + 2M_{11} \end{aligned} \right\} \quad (49)$$

The characteristic equation obtained from the vanishing of the determinant on the left of the matrix equation (48) is

$$\begin{aligned} & \lambda^4 I^2/8 + \lambda^2 \left[\alpha_1 I/2 + \alpha_3 I/4 + I(-J + C - M_{33}) \right] \\ & + \alpha_1 \alpha_3 - (-J + C - M_{33}) \alpha_2 = 0. \end{aligned} \quad (50)$$

This equation, with the help of (49), reduces to

$$\begin{aligned} & \lambda^4 I^2/8 - \lambda^2 \left[(\beta + \alpha - 2M_{11}) I/2 + I/4 (\beta - M) \right] \\ & + (\beta - \alpha - M - M_{33}) (2\alpha + 2M_{33} - M + \beta) = 0, \end{aligned} \quad (51)$$

where

$$\left. \begin{aligned} \beta &= J - D \\ M &= 2M_{11} + M_{33} \end{aligned} \right\}, \quad \alpha = J - C. \quad (52)$$

Rewriting (51) in the form

$$\lambda^4 I^2/8 - \lambda^2 I/4 \left[(5\gamma - 8) \frac{J}{\gamma - 1} + 4C - 4M_{33} \right] + \frac{(3\gamma - 4)}{(\gamma - 1)} (3C - P_{33} - M - M_{33}) J = 0. \quad (53)$$

We have

$$\lambda^2 I^2/4 = 1/4 \left[(5\gamma - 8) \frac{J}{(\gamma - 1)} + 4C - 4M_{33} \right] \pm \left[I^2/16 \left\{ (5\gamma - 8) \frac{J}{(\gamma - 1)} + 4C - 4M_{33} \right\}^2 - \frac{(3\gamma - 4) I^2 J}{2(\gamma - 1)} (3C - P_{33} - M - M_{33}) \right]^{1/2}. \quad (54)$$

Clearly (53) gives a neutral mode

$$\lambda^2 = 0, \text{ for } \gamma = 4/3.$$

The other mode for $\gamma = 4/3$ is

$$\lambda^2 = 8/3I (3C - P_{33} - M - M_{33}) \quad (55)$$

The equation (53) represents the coupling between two modes. It is difficult to resolve these coupled modes in a general case.

Thus we have the following nine modes of oscillations

$$\lambda^2 = 0, \text{ (multiplicity three-neutral mode)}$$

$$\lambda^2 = 4/I (B - A - 4M_{11}), \text{ (multiplicity two-non radial mode).}$$

$$\lambda^2 = 2/I (P - 2M - 3C), \text{ (multiplicity two-non radial mode).}$$

and the other two modes are coupled.

5. Modes of Oscillations of Cylinder when There is no Magnetic Field.

The nine modes of oscillations of the cylinder, when there is no magnetic field can be obtained by putting $M_{11} = M_{33} = 0$ in equations (17) to (25) and proceeding exactly as in last section.

In this case the nine modes are

$$\lambda^2 = 0, \text{ (multiplicity three-neutral mode)}$$

$$\lambda^2 = 4/I (B - A) = 16/15 P/I, \text{ (multiplicity two-non radial mode).}$$

$$\lambda^2 = 4/5 P/I. \text{ (multiplicity two-non radial mode).}$$

and the other two modes are coupled.

For $\gamma = 4/3$, the two coupled modes are

$$\lambda^2 = 0,$$

and

$$\lambda^2 = 32P/45I.$$

For $\gamma < 4/3$, one of the coupled modes becomes unstable.

6. CONCLUDING REMARKS

In this note we have studied the oscillations of gaseous cylinder with and without a magnetic field. It is found that in both the cases we get the same types of modes but of different frequencies. The nine modes of oscillations consist of three groups. The first group is of three neutral modes. The second group of four non-radial modes has two different modes, each of multiplicity two. The third group has two coupled modes.

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RELATION BETWEEN MEIJER-LAPLACE TRANSFORM AND GENERALISED HANKEL TRANSFORM

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ABSTRACT

In the present paper a relation between Meijer-Laplace transform and generalised Hankel transform has been given. Interesting special cases have been discussed and it is illustrated by an example.

1. Meijer-Laplace transform (2, p. 57) of a function $f(t) \in L(0, \infty)$ is defined by the equation

$$(1.1) \quad \phi(s) = \int_0^\infty G_{m, m+1}^{m+1, 0} \left(st \left| \begin{matrix} \eta_1 + \alpha_1, \dots, \eta_m + \alpha_m \\ \eta_1, \dots, \eta_m, \rho \end{matrix} \right. \right) f(t) dt, \quad R(s) > 0.$$

We denote the Meijer-Laplace transform (1.1) symbolically by

$$\phi(s; \alpha_m, \eta_m, \rho) = G[f(t); s; \alpha_m, \eta_m, \rho]$$

where $\phi(s; \alpha_m, \eta_m, \rho) = \phi(s)$, and (1.1) reduces to classical Laplace transform on putting $\alpha_i = 0, i = 1, \dots, m$ and $\rho = 0$. While its generalised Hankel transform (9, p. 298) is defined by the equation.

$$(1.2) \quad g(\xi) = \sqrt{2} \int_0^\infty G_{2p, 2q}^{q, p} \left(\frac{\xi^2 t^2}{4} \left| \begin{matrix} a_1, \dots, a_p, \frac{1}{2} - a_1, \dots, \frac{1}{2} - a_p \\ b_1, \dots, b_q, \frac{1}{2} - b_1, \dots, \frac{1}{2} - b_q \end{matrix} \right. \right) f(t) dt, \quad \xi > 0$$

We denote the generalised Hankel transform (1.2) symbolically by

$$g(\xi; a_p, b_q) = H[f(t); \xi; a_p, b_q]$$

where $g(\xi; a_p, b_q) = g(\xi)$, and (1.2) reduces to Hankel transform of order ν on putting $q = 1, p = 0, b_1 = 1/4 + \nu/2, b_2 = 1/4 - \nu/2$.

The object of this paper is to obtain a relation between the Meijer-Laplace transform of $t^\mu f(t)$ and the generalised Hankel transform of $f(t)$, when $R(\mu) > -1$. The result is stated in the form of a theorem, which yields as particular cases many known and unknown relations, and is then, illustrated by an example.

2. *Theorem:* If $f(t)$ and $H[f(t); \xi; a_p, b_q]$ belong to $L(0, \infty)$ and $R(\mu) > -1, R(s) \geq s_0 > 0, R(\min. \eta_i + 2\min. b_j) > R(-\mu - 1)$ where $i = 1, \dots, m; j = 1, \dots, q$ then

$$G[t^\mu f(t); s; \alpha_m, \eta_m, \rho] = \int_0^\infty k(s, \xi) H[f(t); \xi; a_p, b_q] d\xi.$$

where

$$k(s, \xi) = \frac{1}{\sqrt{\pi}} 2^{\sum_{i=1}^m \eta_i + \rho - \sum_{i=1}^m (\eta_i + \alpha_i) + \mu + \frac{1}{2}} s^{-\mu-1} \\ \times G_{2p+2m+2, 2q+2m}^{q, p+2m+2} \left(\frac{\xi^2}{s^2} \middle| \begin{matrix} * \\ \Delta(2, -\eta_1 - \mu), \dots, \Delta(2, -\eta_m - \mu), \\ \Delta(2, -\eta_1 - \alpha_1 - \mu), \dots, \\ \Delta(2, -\rho - \mu), \frac{1}{2} - {}_1a_p \\ \Delta(2, -\eta_m - \alpha_m - \mu), \frac{1}{2} - {}_1b_q \end{matrix} \right)$$

Proof: Since $f(t)$ and $H[f(t); \xi; a_p, b_q]$ belong to $L(0, \infty)$ and the kernel in (1.2) is a Fourier kernel, therefore using Theorem 2, FOX (5, p. 400), we have

$$f(t) = \sqrt{2} \int_0^\infty H[f(t); \xi; a_p, b_q] G_{2p, 2q}^{q, p} \left(\frac{\xi^2 t^2}{4} \middle| \begin{matrix} {}_1a_p, \frac{1}{2} - {}_1a_p \\ {}_1b_q, \frac{1}{2} - {}_1b_q \end{matrix} \right) d\xi.$$

provided $q \geq p+1$, $R(b_m) \geq (1-h)/4h$, $m=1, \dots, q$;

$R(a_n) < (3h-1)/4h$, $n=1, \dots, p$; $f(t) t^{(1-h)/2h} \in L(0, \infty)$ and $f(t)$ is of bounded variation near $t=\xi$, ($\xi > 0$), where $h=q-p$.

Hence Meijer-Laplace transform of $t^\mu f(t)$ is,

$$(2.1) \quad G[t^\mu f(t); s; \alpha_m, \eta_m, \rho] = \sqrt{2} \int_0^\infty G_{m, m+1}^{m+1, 0} \left(st \middle| \begin{matrix} \eta_1 + \alpha_1, \dots, \eta_m + \alpha_m \\ \eta_1, \dots, \eta_m, \rho \end{matrix} \right) t^\mu \\ \times \int_0^\infty H[f(t); \xi; a_p, b_q] G_{2p, 2q}^{q, p} \left(\frac{\xi^2 t^2}{4} \middle| \begin{matrix} {}_1a_p, \frac{1}{2} - {}_1a_p \\ {}_1b_q, \frac{1}{2} - {}_1b_q \end{matrix} \right) d\xi dt. \\ = \sqrt{2} \int_0^\infty H[f(t); \xi; a_p, b_q] d\xi \\ \times \int_0^\infty t^\mu G_{m, m+1}^{m+1, 0} \left(st \middle| \begin{matrix} \eta_1 + \alpha_1, \dots, \eta_m + \alpha_m \\ \eta_1, \dots, \eta_m, \rho \end{matrix} \right) G_{2p, 2q}^{q, p} \left(\frac{\xi^2 t^2}{4} \middle| \begin{matrix} {}_1a_p, \frac{1}{2} - {}_1a_p \\ {}_1b_q, \frac{1}{2} - {}_1b_q \end{matrix} \right) dt.$$

To justify the change of order of integration in (2.1) we see that ξ -integral is absolutely and uniformly convergent if the generalised Hankel transform of $|g(\xi)|$ exists; the t -integral is absolutely and uniformly convergent, if $p \geq 0$, $q \geq 1$, $(m+q-p) > m > 0$, $R[\min. \eta_i + 2\min. b_j] > R(-\mu-1)$ where $i=1, \dots, m$; $j=1, \dots, q$,

* For the sake of brevity the symbol, ${}_1a_p$ is used to denote a_1, \dots, a_p and $\frac{1}{2} - {}_1a_p$ to denote $\frac{1}{2} - a_1, \dots, \frac{1}{2} - a_p$.

$\neq \Delta(m, \alpha)$ represent the set of parameters

$$\alpha/m, (\alpha+1)/m, \dots, (\alpha+m-1)/m.$$

$|\arg s| < \pi/2$ and $\arg(\xi^2/4)$ may have any value. Hence by Fubini's theorem the inversion of order of integration is justified. The theorem then follows, after evaluating the t -integral with (10, p. 401) and using conditions (6, p. 84), we see

$$\int_0^\infty t^\mu G_{m, m+1}^{m+1, 0} \left(st \left| \begin{matrix} \eta_1 + \alpha_1, \dots, \eta_m + \alpha_m \\ \eta_1, \dots, \eta_m, \rho \end{matrix} \right. \right) G_{2p, 2q}^{q, p} \left(\frac{\xi^2 t^2}{4} \left| \begin{matrix} 1^{ap}, \frac{1}{2} - 1^{ap} \\ 1^{bq}, \frac{1}{2} - 1^{bq} \end{matrix} \right. \right) dt.$$

$$= \frac{1}{\sqrt{\pi}} 2^{\frac{m}{2} \sum_{i=1}^m \eta_i + \rho - \sum_{i=1}^m (\eta_i + \alpha_i) + \mu + \frac{1}{2}} s^{-\mu-1}$$

$$\times G_{2p+2m+2, 2q+2m}^{q, p+2m+2} \left(\frac{\xi^2}{s^2} \left| \begin{matrix} 1^{ap}, \Delta(2, -\eta_1 - \mu), \dots, \Delta(2, -\eta_m - \mu), \Delta(2, -\rho - \mu), \\ 1^{bq}, \Delta(2, -\eta_1 - \alpha_1 - \mu), \dots, \Delta(2, -\eta_m - \alpha_m - \mu), \\ \frac{1}{2} - 1^{ap} \\ \frac{1}{2} - 1^{bq} \end{matrix} \right. \right).$$

3. Various particular cases of this theorem can be obtained, some of which are given here.

(i) Putting $\alpha_i = 0$, $i = 1, \dots, m$; $\rho = \lambda \pm \frac{1}{2}$, $q = 1$, $p = 0$, $b_1 = \frac{1}{2} + \nu/2$, $g(\xi)$ replaced by $\xi^{\frac{1}{2}} g(\xi)$, $f(t)$ by $t^{\frac{1}{2}} f(t)$ and μ by $\mu - \frac{1}{2}$, we get a result due to Verma (12, p. 103).

(ii) Putting $m = 1$, $\alpha_1 = \frac{1}{2} - k - m$, $\eta_1 = 2m$, $\rho = 0$, $q = 1$, $p = 0$, $b_1 = \frac{1}{2} + \nu/2$ and $g(\xi)$ replaced by $\xi^{\frac{1}{2}} g(\xi)$, $f(t)$ by $t^{\frac{1}{2}} f(t)$ and μ by $\mu - \frac{1}{2}$, we obtain another result of Verma (12, p. 105).

(iii) Putting $\rho = 0$, in (i) we get a result due to Bhonsle (4, p. 114).

(iv) Putting $\alpha_i = 0$, $i = 1, \dots, m$, $\rho = 0$, $p = 1$, $q = 2$, $a_1 = k - m - \nu/2 - \frac{1}{2}$, $b_1 = \frac{1}{2} + \nu/2$, $b_2 = \nu/2 + 2m + \frac{1}{2}$, we get a relation between Laplace transform and $X_{\nu, k, m}$ -transform (8, p. 271).

$$\propto |t^\mu f(t); s| = \int_0^\infty k(s, \xi) x_{\nu, k, m} [f(t); \xi] d\xi.$$

where

$$k(s, \xi) = \frac{2^{\mu + \frac{1}{2}} s^{-\mu-1}}{\sqrt{\pi}} G_{44}^{23} \left(\xi^2/s^2 \left| \begin{matrix} k - m - \nu/2 - \frac{1}{2}, -\mu/2, (-\mu+1)/2, -k + m + \nu/2 + \frac{3}{2} \\ \frac{1}{2} + \nu/2, \nu/2 + 2m + \frac{1}{2}, \frac{1}{2} - \nu/2, \frac{1}{2} - \nu/2 - 2m \end{matrix} \right. \right)$$

provided $X_{\nu, k, m}$ -transform of $|g(\xi)|$ exists, $R(\mu) > -1$, $R(s) \geq s_0 > 0$, $R(\nu + 3/2 + m \pm m) > 0$ and $2m$ is not an integer or zero.

(v) Putting $\alpha_i = 0, i = 1, \dots, m-1, \alpha_m = -\delta - \lambda, \eta_m = \delta - \lambda, \rho = -\delta - \lambda, q = 1, p = 0, b_1 = 1/4 + v/2$, we get a relation (11, p. 357) between Meijer's transform (7, p. 727) and Hankel transform of order v .

$$\psi(s) = \int_0^\infty k(s, \xi) H_v[f(t); \xi] d\xi.$$

Where $\psi(s)$ denotes Meijer's transform of $t^\mu f(t)$ and

$$k(s, \xi) = \frac{2^{\mu + \frac{1}{2}} s^{-\mu-1}}{\sqrt{\pi}} G_{44}^{14} \left(\frac{\xi^2}{s^2} \middle| \begin{matrix} \Delta(2, \lambda - \delta - \mu), \Delta(2, \delta + \lambda - \mu) \\ 1/4 + v/2, \Delta(2, 2\lambda - \mu), 1/4 - v/2 \end{matrix} \right)$$

provided $H_v \in L(0, \infty), R(\mu) > -1, R(s) \geq s_0 > 0$ and $R(\mu - \lambda + v + 3/2 \pm \delta) > 0$.

4. To illustrate the theorem let us take $f(t) = E_v(t)$, (3, p. 157)

$$\text{where } E_v(t) = e^{-\frac{1}{2}t} W_{-v/2, \frac{1}{2}-v/2}(t) t^{v/2-1}.$$

using (1, p. 216), we get,

$$E_v(t) = G_{12}^{20} \left(t \middle| \begin{matrix} v \\ 0, v-1 \end{matrix} \right)$$

Then from (6, p. 84), we have

$$G \left[t^\mu E_v(t); s; \alpha_m, \eta_m, \rho \right] = G_{m+2, m+2}^{m+1, 2} \left(s \middle| \begin{matrix} -\mu, 1-\mu-v, \eta_1 + \alpha_1, \dots, \eta_m + \alpha_m \\ \eta_1, \dots, \eta_m, \rho, -\mu-v \end{matrix} \right)$$

where $m > 0, R(\eta_i, \rho + \mu) > -1, R(\eta_i, \rho + \mu + v - 1) > -1$, and $|\arg s| < \pi/2$.

Now using (10, p. 401), we get

$$H[E_v(t); \xi; a_p, b_q] = \frac{1}{\sqrt{2\pi}} G_{2p+4, 2q+2}^{q, p+4} \left(\frac{\xi^2}{s^2} \middle| \begin{matrix} 1^{ap}, 0, \frac{1}{2}, (1-v)/2, (2-v)/2, \frac{1}{2} - 1^{ap} \\ 1^{bq}, -v/2, (1-v)/2, \frac{1}{2} - 1^{bq} \end{matrix} \right)$$

with $p \geq 0, q \geq 1, R(1 + 2 \min. b_j) > 0, R(v + 2 \min. b_j) > 0, j = 1, \dots, q$.

substituting this in the theorem, we obtain the result using (6, p. 84)

$$\frac{2^{\mu + \rho - \sum_{i=1}^m \alpha_i} s^{-\mu-1}}{\pi} \int_0^\infty G_{2p+2m+2, 2q+2m}^{q, p+2m+2}$$

$$\begin{aligned}
& \left(\xi^2/s^2 \left| \begin{array}{l} {}_1a_p, \Delta(2, -\eta_1 - \alpha_1), \dots, \Delta(2, -\eta_m - \alpha_m), \Delta(2, -\rho - \mu), \frac{1}{2} - {}_1a_p \\ {}_1b_q, \Delta(2, -\eta_1 - \alpha_1 - \mu), \dots, \Delta(2, -\eta_m - \alpha_m - \mu), \frac{1}{2} - {}_1b_q \end{array} \right. \right) \\
& \times G_{2p+4, 2q+2}^{q, p+4} \left(\xi^2 \left| \begin{array}{l} {}_1a_p, 0, \frac{1}{2}, (1-v)/2, (2-v)/2, \frac{1}{2} - {}_1a_p \\ {}_1b_q, -v/2, (1-v)/2, \frac{1}{2} - {}_1b_q \end{array} \right. \right) d\xi \\
(4.1) = & \frac{2^{\mu+\rho-\sum_1^m \alpha_i}}{2\pi} G_{2m+4, 2m+4}^{2m+2, 4} \left(s^2 \left| \begin{array}{l} -\mu/2, -\mu/2+\frac{1}{2}, -\mu/2-v/2+\frac{1}{2}, \\ \Delta(2, \eta_1), \dots, \Delta(2, \eta_m) \Delta(2, \rho), \\ -\mu/2-v/2+1, \Delta(2, \eta_1+\alpha_1), \dots, \Delta(2, \eta_m+\alpha_m) \\ -\mu/2-v/2, -\mu/2-v/2+\frac{1}{2} \end{array} \right. \right)
\end{aligned}$$

with $m+q-p > m > 0$, $p \geq 0$, $q \geq 1$, $|\arg s| < \pi/2$, $R(\mu) > -1$, $R(2b_j - \frac{1}{2}) > -1$,
 $R(2a_j - \frac{1}{2}) < 1$, $R(-\eta/2 - \mu/2 + a_j) < 1$, $R(-\rho/2 - \mu/2 + a_i) < 1$, $R(a_i) < 1$,
 $R(a_j + 1/2 - v/2) < 1$,
 $R[(-\eta_i - \mu + 1)/2] < 1$, $R[(-\eta_1 - \mu - v + 2)/2] < 1$, $R[(-\rho - \mu + 1)/2] < 1$,
 $R[(-\rho - \mu - v + 2)/2] < 1$,
 $j=1, \dots, p$; $i=1, \dots, m$.

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9	143	15	$\left[-\frac{\alpha}{\xi_o^2} - \frac{2(q+)}{v \xi_o^{2q+5}} \right] a_3$	$\left[-\frac{\alpha}{\xi_o^2} - \frac{2(q+1)S}{v \xi_o^{2q+5}} \right] a_3$
10	143	Last line	$\left[\frac{4\pi K^2 G}{3(p-3)R^{2p-2}} \cdot \frac{p\kappa^{p-3} - 3\mu^{p-3}}{\kappa^{2q-2/p-3}} \right]_{\kappa=1}$	$\left[\frac{4\pi k^2 G}{3(p-3)R^{2p-2}} \cdot \frac{p\kappa^{p-3} - 3\mu^{p-3}}{\kappa^{2p-2}\mu^{p-3}} \right]$
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